Math 538: Problem Set 2

Do a good amount of problems; choose problems based on what you already know and what you need to practice. Examples are important.

Ideals and primes

Problems 1-2 are good for you, but not essential for submission.

Fix a ring R

1. (The product operation on ideals)
   (a) Show that the operation of multiplication of ideals in R is commutative and associative, and has the identity element R. If R is an integral domain show that $IJ \neq (0)$ unless $I = (0)$ or $J = (0)$.
   (b) Now let P, I, J be ideals of R with P prime and $P \supset IJ$. Show that $P \supset I$ or $P \supset J$.
   (c) Extend (b) to the caes of $P \supset \prod_{i=1}^{n} I_i$ where I_i is a finite set of ideals.

2. (Review: The CRT)
   (a) Let $I_1, \ldots, I_k$ be ideals, and suppose that $I_i + I_j = R$ where $i \neq j$. Show that $\prod_{i=1}^{k} I_i = \bigcap_{i=1}^{k} I_i$ and that the natural map $R \big/ \cap_{i=1}^{k} I_i \to \oplus_{i=1}^{k} R/I_i$ is a well-defined isomorphism of rings.
   (b) Suppose that $\sum_{i=1}^{k} I_i = R$, and let $n_i \in \mathbb{Z}_{\geq 1}$. Show that $\sum_{i=1}^{k} I_i^{n_i} = R$.

Unique factorization

Let L/K be an extension of number fields.

3. (Primes above and below)
   (a) Let $\mathfrak{A} \lhd \mathcal{O}_L$ be a non-zero proper ideal. Show that $\mathfrak{A} \cap K = \mathfrak{A} \cap \mathcal{O}_K$ and that this is a non-zero proper ideal of $\mathcal{O}_K$.
   (b) Let $\mathfrak{P} \lhd \mathcal{O}_L$ be a non-zero prime ideal. Show that $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_K$ is a prime ideal of $\mathcal{O}_K$. We say that $\mathfrak{P}$*lies above* $\mathfrak{p}$ and write $\mathfrak{P}|\mathfrak{p}$.
   (c) Show that $\mathfrak{P}$lies above $\mathfrak{p}$ iff $\mathfrak{P}|\mathfrak{p}\mathcal{O}_L$ as ideals of $\mathcal{O}_L$.

4. The map $\mathfrak{a} \mapsto \mathfrak{a}\mathcal{O}_K$.
   (a) Let $\mathfrak{a} \lhd \mathcal{O}_K$ be a proper ideal. Show that there is $\gamma \in K \setminus \mathcal{O}_K$ such that $\gamma\mathfrak{a} \subset \mathcal{O}_K$, and conclude that $\mathfrak{a}\mathcal{O}_L$ is a proper ideal of $\mathcal{O}_L$.
   (a) Let $\mathfrak{a}, \mathfrak{b}$ be (fractional) ideals of $\mathcal{O}_K$. Show that $(\mathfrak{a}\mathfrak{b})\,\mathcal{O}_L = (\mathfrak{a}\mathcal{O}_L)\,(\mathfrak{b}\mathcal{O}_L)$.
   (b) Let $\mathfrak{a}, \mathfrak{b}$ be ideals of $\mathcal{O}_K$. Comparing prime factorizations in $\mathcal{O}_K, \mathcal{O}_L$ show that $\mathfrak{a}\mathcal{O}_L | \mathfrak{b}\mathcal{O}_L \Rightarrow \mathfrak{a}|\mathfrak{b}$. Conclude that the map $\mathfrak{a} \to \mathfrak{a}\mathcal{O}_L$ is injective on fractional ideals.
   (c) Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$. Show that $\mathfrak{a}\mathcal{O}_L \cap \mathcal{O}_K = \mathfrak{a}$.

An example

5. (Dedekind) Let $K = \mathbb{Q}(\theta)$ where θ is a root of $f(x) = x^3 - x^2 - 2x - 8$.
   (a) Show that f is irreducible over $\mathbb{Q}$.
   (b) Verify that $\eta = \frac{\theta^2+\theta}{2}$ satisfies $\eta^3 - 3\eta^2 - 10\eta - 8 = 0$.
   (c) Show that $1, \theta, \eta$ are linearly independent over $\mathbb{Q}$.
   (d) Let $M = \mathbb{Z} \oplus \mathbb{Z}\theta \oplus \mathbb{Z}\eta$ and let $N = \mathbb{Z}[\theta] = \mathbb{Z} \oplus \mathbb{Z}\theta \oplus \mathbb{Z}\theta^2$. Show that $N \subset M \subset \mathcal{O}_K$.
   (f) Calculate $D_{K/\mathbb{Q}}(M)$, $D_{K/\mathbb{Q}}(N)$ (hint: $D_{K/\mathbb{Q}}(N) = -4 \cdot 503$).

(h) Show that $M = \mathcal{O}_K$

Hint: Let $\{\alpha, \beta, \gamma\}$ be an integral basis and consider $\frac{d_{K/\mathbb{Q}}(1,\theta,\eta)}{d_{K/\mathbb{Q}}(\alpha,\beta,\gamma)}$.

(i) Let $\delta = A + B\theta + C\eta$ with $A, B, C \in \mathbb{Z}$. Show that $2 | d_{K/\mathbb{Q}}(\mathbb{Z}[\delta])$, and conclude that $\mathbb{Z}[\delta] \neq \mathcal{O}_K$.

Localization

6. (Localization at a prime) Let $\mathfrak{p} \lhd \mathcal{O}_K$ be a prime of K.

(a) Show that $\mathcal{O}_{K,\mathfrak{p}} = \left\{ \frac{\alpha}{s} \mid \alpha, s \in \mathcal{O}_K,\, s \notin \mathfrak{p} \right\}$ is a subring of K. It is called the *localization* of $\mathcal{O}_K$ at $\mathfrak{p}$.

(b) Show that $\mathfrak{p}\mathcal{O}_{K,\mathfrak{p}}$ is an ideal in $\mathcal{O}_{K,\mathfrak{p}}$, and that its complement consists of $(\mathcal{O}_{K,\mathfrak{p}})^{\times}$. Conclude that $\mathcal{O}_{K,\mathfrak{p}}$ is a *local ring*: it has a unique maximal ideal.

RMK The localization of any ring at a prime ideal is a local ring.

(c) Let $x \in K^{\times}$. By considering the prime factorization of the fractional ideal (x) show that at least one of $x, x^{-1} \in \mathcal{O}_{K,\mathfrak{p}}$.

DEF A subring of a field satisfying the property of (c) is called a *valuation ring.*

(**d) Show that every ideal of $\mathcal{O}_{K,\mathfrak{p}}$ is of the form $(\mathfrak{p}\mathcal{O}_{K,\mathfrak{p}})^k$ for some $k \geq 0$.

(e) Let L be a finite extension of K. Show that $\left\{ \frac{x}{s} \mid x \in \mathcal{O}_L,\, s \in \mathcal{O}_K \setminus \mathfrak{p} \right\}$ is a subring of L (the localization of $\mathcal{O}_L$ at $\mathfrak{p}$), finitely generated as an $\mathcal{O}_{K,\mathfrak{p}}$-module, and use the structure theory of modules over a PID to conclude that it is of the form $\mathcal{O}_{K,\mathfrak{p}}^n$ where $n = [L : K]$.

Completion

7. Let (X, d) be a metric space, $\left(\hat{X}, \hat{d}\right)$ its metric completion. Let (Y, d_Y) be a complete metric space.

(a) Let $f \colon X^n \to Y$ be *uniformly continuous on balls*: Given $z \in X$ and $\varepsilon, R > 0$ there is $\delta > 0$ such that if $\underline{x}, \underline{x}' \in \mathbb{R}^n$ satisfy for all i that $d(x_i, x_i') < \delta$ for $d(z, x_i), d(z, x_i') \leq R$ then

$$d_Y\left(f(\underline{x}), f(\underline{x}')\right) < \varepsilon\,.$$

Show that f extends uniquely to a continuous function $\hat{f} \colon \hat{X}^n \to Y$.

(b) Suppse that X is also a group, and that the map $(x, x') \to x^{-1}x'$ is uniformly continuous. Show that $\hat{X}$ has a unique group structure continuosly extending that of X.

(c) In the setting of B, let $H < X$ be a subgroup. Show that the closure of H in $\hat{X}$ is a subgroup, and that if H is normal in X then the closure is normal in $\hat{X}$.