MATH 253 – PROJECTIONS

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The three definitions

Suppose we have a vector $\vec{w}$ we'd like to *project along a vector* $\vec{v}$. In other words, we'd like to find the *component* of $\vec{w}$ *in the direction* of $\vec{v}$. We then defined three quantities:

(1) The "scalar projection of $\vec{w}$ along $\vec{v}$" is the *number*

$$\frac{\vec{w}\cdot\vec{v}}{|\vec{v}|} = \vec{w}\cdot\left(\frac{1}{|\vec{v}|}\vec{v}\right).$$

It measures the *magnitude* of the component of $\vec{w}$ along $\vec{v}$, and ought to be called that.

(2) The "vector projection of $\vec{w}$ along $\vec{v}$" is the *vector*

$$\frac{\vec{w}\cdot\vec{v}}{|\vec{v}|}\cdot\frac{\vec{v}}{|\vec{v}|} = \left(\frac{\vec{w}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v},$$

having magnitude as in (1) and direction along $\vec{v}$. We will also call it "the component of $\vec{w}$ along $\vec{v}$" or "the component of $\vec{w}$ in the direction of $\vec{v}$".

(3) The misnamed "orthogonal projection", that being the remainder vector

$$\vec{w} - \left(\frac{\vec{w}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}.$$

[please don't use this term outside this course; you should really call it "the component of $\vec{w}$ orthogonal to $\vec{v}$"].

Side Calculations

We also did in class a little calculation, to verify that what we just called "the component orthogonal to $\vec{v}$" really is orthogonal to $\vec{v}$:

$$\begin{aligned}\left[\vec{w} - \left(\frac{\vec{w}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}\right]\cdot\vec{v} &= \vec{w}\cdot\vec{v} - \left(\frac{\vec{w}\cdot\vec{v}}{|\vec{v}|^2}\right)(\vec{v}\cdot\vec{v}) \\ &= \vec{w}\cdot\vec{v} - \left(\frac{\vec{w}\cdot\vec{v}}{|\vec{v}|^2}\right)|\vec{v}|^2 \\ &= 0\end{aligned}$$

where in the second line we used that $\vec{v}\cdot\vec{v} = |\vec{v}|^2$.

Finally, one can do the calculation to check that "the component of $\vec{w}$ along $\vec{v}$" and "the component of $\vec{w}$ orthogonal to$\vec{v}$" together add up to $\vec{w}$:

$$\left(\frac{\vec{w}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} + \left[\vec{w} - \left(\frac{\vec{w}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}\right] = \vec{w} + \left[\left(\frac{\vec{w}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} - \left(\frac{\vec{w}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}\right] = \vec{w}.$$

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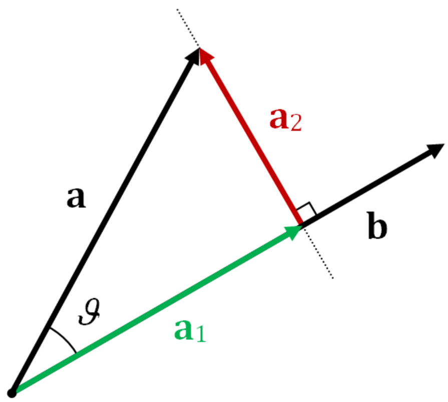
Geometric Picture

To see what decomposing a vector into components along and orthogonal to another vector, see the following picture:



Here the vector $\vec{a}$ is projected along the vector $\vec{b}$. $\vec{a}_1$ is the component along $\vec{b}$, $\vec{a}_2$ is the component orthogonal to $\vec{b}$.

(Image credit: user Paolo.dL on Wikipedia; see `http://en.wikipedia.org/wiki/File:Projection_and_rejection.png`.