MATH 253 – WORKSHEET 4
EQUATIONS OF LINES AND PLANES

Reminder: $\vec{C} = \vec{A} \times \vec{B}$ has magnitude $\left|\vec{C}\right| = \left|\vec{A}\right| \left|\vec{B}\right| \sin\theta$, direction perpendicular to $\vec{A}, \vec{B}$ so that the $\vec{A}, \vec{B}, \vec{C}$ is right-handed in that order. In coordinates

$$\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \begin{vmatrix} \vec{i} & j & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

1. Working on a planes

(1) We will find a unit vector normal to the plane passing through the points $(3, 0, 0)$, $(0, 2, 0)$, $(0, 0, 4)$. (normal = perpendicular; unit = magnitude 1)

(a) Find two vectors parallel to the plane:
Solution: (there are many)
$\vec{A} = (0, 0, 4) - (3, 0, 0) = \langle -3, 0, 4 \rangle$, $\vec{B} = (0, 2, 0) - (3, 0, 0) = \langle -3, 2, 0 \rangle$.

(b) Find their cross product
Solution: $\vec{A} \times \vec{B} = \langle 0 \cdot 0 - 4 \cdot 2, 4 \cdot (-3) - (-3) \cdot 0, (-3) \cdot 2 - 0 \cdot (-3) \rangle = -\langle 8, 12, 6 \rangle = (-2) \langle 4, 6, 3 \rangle$.

(c) *Normalize* to obtain a unit vector.
Solution: We first rescale by (-2) and the divide by the norm $\sqrt{4^2 + 6^2 + 3^2} = \sqrt{61}$ to find the vector $\frac{1}{\sqrt{61}} \langle 4, 6, 3 \rangle$.

2. Lines and PLanes

(1) Find equations for the line through $(2, 0, 3)$, $(3, 4, 0)$.
Solution: The vector $\vec{v} = \langle 1, 4, -3 \rangle$ is parallel to the line, so the equations are

$$\frac{x - 2}{1} = \frac{y}{4} = \frac{z - 3}{-3}$$

or, equivalently

$$\begin{cases} y = 4x - 8 \\ z = -3x + 9 \end{cases}.$$

(2) Find an equation for the plane passing through $(3, 0, 0)$, $(0, 2, 0)$, $(0, 0, 4)$.
Solution 1: In a previous problem we found that $\vec{N} = \langle 4, 6, 3 \rangle$ is normal to this plane. It follows that a general point (x, y, z) on the plane satisfies

$$\langle x - 3, y - 0, z - 0 \rangle \cdot \langle 4, 6, 3 \rangle = 0$$

or equivalently, that

$$4x + 6y + 3z = 12\,.$$

Solution 2: (Brute force) We need numbers a, b, c, d such that for every point in the plane, $ax + by + cz = d$. Guessing that $d \neq 0$, we could divide the equation by d, so it is enough to find a, b, c such that

$$ax + by + cz = 1$$

for the whole plane, and in particular for the three given points. So we must have

$$\begin{cases} a \cdot 3 + b \cdot 0 + c \cdot 0 &= 1 \\ a \cdot 0 + b \cdot 2 + c \cdot 0 &= 1 \\ a \cdot 0 + b \cdot 0 + c \cdot 4 &= 1 \end{cases}.$$

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This is an easy 3×3 system (in general it would be harder), and has the solution

$$a = \frac{1}{3}; \qquad b = \frac{1}{2}; \qquad c = \frac{1}{4}$$

and we get the equation

$$\frac{1}{3}x + \frac{1}{2}y + \frac{1}{4}z = 1$$

(which is the same as the one from the previous solution up to an overall factor of 12).