MATH 253 – WORKSHEET 8
PARTIAL DERIVATIVES

1. Differentiate the following functions

(1) $f(x,y) = \frac{y}{x^2+y^2}$

(a) $f_x = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) \overset{\text{y is const}}{=} y\frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2}\right) \overset{\text{quot rule}}{=} y\frac{-2x}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$

(b) $f_y = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) \overset{\text{quot rule}}{=} \frac{\left(\frac{\partial}{\partial y}y\right)\left(x^2+y^2\right)-y\frac{\partial}{\partial y}\left(x^2+y^2\right)}{(x^2+y^2)^2} = \frac{x^2+y^2-y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

(2) Let $z = \sqrt{1-x^2-y^2}$.

(a) $\frac{\partial z}{\partial x} \overset{\text{chain rule}}{=} \frac{1}{2}\frac{\frac{\partial}{\partial x}\left(1-x^2-y^2\right)}{\sqrt{1-x^2-y^2}} = -\frac{x}{\sqrt{1-x^2-y^2}}$

(b) Use $\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right) = \frac{\partial}{\partial x}(1) = 0$ to find $\frac{\partial z}{\partial x}$ a different way:
Solution: $0 = \frac{\partial}{\partial x}(1) = \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right) = 2x + 0 + 2z\cdot\frac{\partial z}{\partial x}$ (explanation: when we take the partial derivative by x, we have $\frac{\partial}{\partial x}x^2 = 2x$, we have that y is constant, and we differentiate z using the chain rule, noting that z depends on x). Solving the equation for $\frac{\partial z}{\partial x}$ we have

$$\frac{\partial z}{\partial x} = -\frac{2x}{2z} = -\frac{x}{z} = -\frac{x}{\sqrt{1-x^2-y^2}}\,.$$

(3) $g(x,y) = \ln(x^2+y^2)$

(a) $g_x = \frac{2x}{x^2+y^2}$ and by symmetry $g_y = \frac{2y}{x^2+y^2}$.

(b) $g_{xx} = \frac{2\left(x^2+y^2\right)-2x(2x)}{(x^2+y^2)^2} = 2\frac{x^2-y^2}{(x^2+y^2)^2}$ while $g_{xy} = 2x\cdot\frac{-2y}{(x^2+y^2)^2} = -\frac{4xy}{(x^2+y^2)^2}$.

(c) $g_{yx} = -\frac{4xy}{(x^2+y^2)^2}$ by problem 1(a) while $g_{yy} = 2\frac{y^2-x^2}{(x^2+y^2)^2}$ by symmetry (reverse the roles of x, y in part (b)).

(d) $\Delta g = g_{xx} + g_{yy} = 0$.

Remark. $g(x,y)$ is the electric potential in two dimensions.

Date: 23/9/2013.