MATH 253 – WORKSHEET 10
TANGENT PLANES

Find the equation of the plane tangent to the following surfaces at the following points:

(1) $z = e^{-x^2-y^2}$ at $(0,0,1)$.

Solution: $\frac{\partial z}{\partial x} = -2xe^{-x^2-y^2}$, $\frac{\partial z}{\partial y} = -2ye^{-x^2-y^2}$, $\frac{\partial z}{\partial x}(0,0) = 0$, $\frac{\partial z}{\partial y}(0,0) = 0$, and the equation is

$$z = 1\,.$$

(2) $z = e^{-x^2-y^2}$ at $\left(1,1,e^{-2}\right)$.

Solution: Now $\frac{\partial z}{\partial x}(1,1) = -2e^{-2}$, $\frac{\partial z}{\partial y}(1,1) = -2e^{-2}$, and the equation is

$$z - e^{-2} = -2e^{-2}(x-1) - 2e^{-2}(y-1)$$

which we can rearrange to

$$2x + 2y + e^2 z = 5\,.$$

(3) Two planes tangent to the surface $z = 1 - x^2 - y^2$ meet the x-axis at $\frac{21}{16}$ and the y-axis at $\frac{21}{8}$. What are they? Where are the points of tangency?

Solution: Suppose the point of tangency is (x_0, y_0, z_0). The equation of the tangent plane is

$$z = z_0 - 2x_0(x - x_0) - 2y_0\,(y - y_0)$$

that is

$$\begin{aligned} 2x_0 x + 2y_0 y + z &= 1 - x_0^2 - y_0^2 + 2x_0^2 + 2y_0^2 \\ &= 1 + x_0^2 + y_0^2\,. \end{aligned}$$

We are given that this plane contains $\left(\frac{21}{16}, 0, 0\right)$ and $\left(0, \frac{21}{8}, 0\right)$, that is that:

$$\begin{cases} 2x_0\frac{21}{16} & = 1 + x_0^2 + y_0^2 \\ 2y_0\frac{21}{8} & = 1 + x_0^2 + y_0^2 \end{cases}.$$

In particular we have $\frac{21}{8}x_0 = \frac{21}{4}y_0$ so $x_0 = 2y_0$. Plugging into the first equation we find $\frac{21}{4}y_0 = 1 + 5y_0^2$, that is

$$5y_0^2 - \frac{21}{4}y_0 + 1 = 0\,.$$

By the quadratic formula,

$$y_0 = \frac{\frac{21}{4} \pm \sqrt{\frac{21^2}{4^2} - 20}}{10} = \frac{21 \pm \sqrt{441 - 16\cdot 20}}{40} = \frac{21 \pm \sqrt{121}}{40} = \frac{21 \pm 11}{40}$$

so the two possibilities are $y_0 = \frac{32}{40} = \frac{4}{5}$ and $y_0 = \frac{10}{40} = \frac{1}{4}$. The points of tangency are therefore $\left(\frac{8}{5}, \frac{4}{5}, -\frac{11}{5}\right)$ where the plane is $\frac{16}{5}x + \frac{8}{5}y + z = \frac{21}{5}$ or $16x + 8y + 5z = 21$ and $\left(\frac{1}{2}, \frac{1}{4}, \frac{11}{16}\right)$ where the plane is $x + \frac{1}{2}y + z = \frac{21}{16}$.

Date: 27/9/2013.