MATH 253 – WORKSHEET 11
LINEAR APPROXIMATION

Use a linear approximation to estimate

(1) e^{x-y} at $(0.1, 0.1)$.

Solution: We will expand $f(x,y) = e^{x-y}$ about the point $(0,0)$. We have $f(0,0) = 1$, $\frac{\partial f}{\partial x}(0,0) = e^{x-y}\restriction_{(0,0)} = 1$, $\frac{\partial f}{\partial y}(0,0) = -e^{-x-y}\restriction_{(0,0)} = -1$ so the linear approximation is $f(x,y) \approx 1 + x - y$, and $f(0.1, 0.1) \approx 1 + 0.1 - 0.1 = 1$.

Remark: From first year we know that, to first order, $e^x \approx 1 + x$, $e^{-y} \approx 1 - y$. Multiplying the two we get $e^{x-y} \approx (1+x)(1-y) = 1 + x - y - xy \approx 1 + x - y$ to first order (i.e. neglecting the second-order term xy).

(2) The area of a triangle two of whose sides are 9.9cm and 10.1cm long and meet at an angle of $31°$.

Solution: Say the sides have lengths a, b and the angle between them is θ. Then the area has the form

$$A = \frac{1}{2} ab \sin\theta \,.$$

Now expand about $(10, 10, 30°)$. We have $\frac{\partial A}{\partial a} = \frac{1}{2} b \sin\theta$, $\frac{\partial A}{\partial b} = \frac{1}{2} a \sin\theta$, $\frac{\partial A}{\partial \theta} = \frac{1}{2} ab \cos\theta$. Thus $A(10, 10, 30°) = 25$, $\frac{\partial A}{\partial a}(10, 10, 30°) = \frac{5}{2} = \frac{\partial A}{\partial b}(10, 10, 30°)$ and $\frac{\partial A}{\partial \theta}(10, 10, 30°) = 25\sqrt{3}$ (since $\cos 30° = \frac{\sqrt{3}}{2}$). The linear approximation is then

$$A\,(9.9, 10.1, 31°) \approx 25 + \frac{5}{2}\Delta x + \frac{5}{2}\Delta y + 25\sqrt{3}\Delta\theta$$

where $\Delta x = 9.9 - 10 = -0.1$, $\Delta y = 10.1 - 10 = 0.1$ and $\Delta\theta = 1° = \frac{\pi}{180}$rad (note that $\frac{d \sin\theta}{d\theta} = \cos\theta$ only if θ is measured in radians!). The final answer is therefore

$$\begin{aligned} A &\approx 25 - \frac{5}{20} + \frac{5}{20} + \frac{25\sqrt{3}\pi}{180} \\ &= 25 + \frac{5\sqrt{3}\pi}{36} \\ &\approx \text{(with a calculator) } 25 + 0.756 \end{aligned}$$

(3) An equilateral triangle has sides of length 10cm (to within 1mm) and the angle between those sides is $150°$ (to within $1°$). What is the area of the triangle? Estimate the maximum error in your calculation.

Solution: As in the previous problem we have $dA = \left(\frac{1}{2} b \sin\theta\right) dx + \left(\frac{1}{2} a \sin\theta\right) dy + \left(\frac{1}{2} ab \cos\theta\right) d\theta$. Plugging in the values for a, b, θ and $\Delta x, \Delta y, \Delta\theta$ the error estimate is

$$\begin{aligned} |\Delta A| &\lesssim \left|\frac{5}{2}\cdot 0.1\right| + \left|\frac{5}{2}\cdot 0.1\right| + \left|\left(-25\sqrt{3}\right)\cdot\frac{\pi}{180}\right| \\ &= \frac{1}{2} + \frac{5\sqrt{3}\pi}{36} \approx 1.256\,. \end{aligned}$$

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(4) Two planes tangent to the surface $z = 1 - x^2 - y^2$ meet the x-axis at $\frac{21}{16}$ and the y-axis at $\frac{21}{8}$. What are they? Where are the points of tangency?

Solution:

(a) Interpretation: we are given the surface $z = 1 - x^2 - y^2$, and need to find planes which are (1) tangent to the surface (2) pass through $\left(\frac{21}{16}, 0, 0\right)$ and $\left(0, \frac{21}{8}, 0\right)$.

(b) Parametrization: Our unknowns will be a, b such that $\left(a, b, 1 - a^2 - b^2\right)$ is the point of tangency of the plane to the curve.

(c) Interpretation II: We implement condition (1) by finding the tangent plane at this point. The partial derivatives are $\frac{\partial z}{\partial x}(a,b) = -2x \restriction_{(a,b)} = -2a$, $\frac{\partial z}{\partial y} \restriction_{(a,b)} = -2y \restriction_{(a,b)} = -2b$ so the equation of the tangent plane is

$$z - \left(1 - a^2 - b^2\right) = (-2a)(x - a) + (-2b)(y - b)$$

or equivalently

$$z + 2ax + 2by = 1 - a^2 - b^2 + 2a^2 + 2b^2$$

that is

$$z + 2(ax + by) = 1 + a^2 + b^2 .$$

We implement condition (2) by insisting that this plane passes through $\left(\frac{21}{16}, 0, 0\right)$ and $\left(0, \frac{21}{8}, 0\right)$, that is that

$$\begin{cases} 0 + 2\left(a\frac{21}{16} + 0\right) & = 1 + a^2 + b^2 \\ 0 + 2\left(0 + b\frac{21}{8}\right) & = 1 + a^2 + b^2 \end{cases}$$

so, conditions (1),(2) amount to the system of equations

$$\begin{cases} \frac{21}{8}a & = 1 + a^2 + b^2 \\ \frac{21}{4}b & = 1 + a^2 + b^2 \end{cases} .$$

(d) Solving the equations: Combining the equations we find $\frac{21}{8}a = \frac{21}{4}b$ so $a = 2b$. Plugging into the second equation we find

$$\frac{21}{4}b = 1 + (2b)^2 + b^2 ,$$

that is

$$5b^2 - \frac{21}{4}b + 1 = 0 .$$

It follows that

$$\begin{aligned} b &= \frac{\frac{21}{4} \pm \sqrt{\frac{441}{16} - 20}}{10} \\ &= \frac{21}{40} \pm \frac{1}{40}\sqrt{441 - 16 \cdot 20} \\ &= \frac{21}{40} \pm \frac{\sqrt{121}}{40} \\ &= \frac{21 \pm 11}{40} . \end{aligned}$$

In other words, the two solutions are $b = \frac{32}{40} = \frac{4}{5}$, $a = \frac{8}{5}$ and $b = \frac{10}{40} = \frac{1}{4}$, $a = \frac{1}{2}$.

(e) Endgame:

(i) At the point of tangency $\left(\frac{8}{5}, \frac{4}{5}, 1 - \frac{16+64}{25}\right)$ we have the plane $z + \frac{16}{5}x + \frac{8}{5}y = \frac{105}{25} = \frac{21}{5}$.

(ii) At the point of tangency $\left(\frac{1}{4}, \frac{1}{2}, 1 - \frac{1}{16} + \frac{1}{4}\right)$ we have the plane $z + \frac{1}{2}x + y = \frac{21}{16}$.