MATH 253 – WORKSHEET 13
DIRECTIONAL DERIVATIVES

(1) In each case find $\vec{\nabla} f$ and $D_{\vec{u}} f$ at the given point.

(a) $f(x,y) = xe^y$, at $(1,0)$ in the direction $\vec{u} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$.

Solution: $\vec{\nabla} f(x,y) = \langle e^y, xe^y \rangle$ so $\vec{\nabla} f(1,0) = \langle 1,1 \rangle$ and

$$\begin{aligned} D_{\left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle} f(1,0) &= \langle 1,1 \rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle \\ &= \frac{3}{5} - \frac{4}{5} = -\frac{1}{5}. \end{aligned}$$

(b) $f(x,y,z) = x^2 + y^2 + z^2$ at $(1,2,3)$ in the direction $\vec{u} = \left\langle -\frac{6}{11}, \frac{7}{11}, \frac{6}{11} \right\rangle$.

Solution: $\vec{\nabla} f(x,y,z) = \langle 2x, 2y, 2z \rangle$ so $\vec{\nabla} f(1,2,3) = 2\langle 1,2,3 \rangle$ and

$$\begin{aligned} D_{\left\langle -\frac{6}{11}, \frac{7}{11}, \frac{6}{11} \right\rangle} f(1,2,3) &= 2\langle 1,2,3 \rangle \cdot \left\langle -\frac{6}{11}, \frac{7}{11}, \frac{6}{11} \right\rangle \\ &= \frac{2}{11} \langle -6 + 14 + 18 \rangle = \frac{52}{11}. \end{aligned}$$

(c) $f(x,y) = \sqrt{x^2+y^2} + e^x$ at $(1,1)$ in the direction making an angle $\frac{\pi}{4}$ to the horizontal.

Solution: $\vec{\nabla} f(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}} + e^x, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$ so $\vec{\nabla} f(1,1) = \left\langle \frac{1}{\sqrt{2}} + e, \frac{1}{\sqrt{2}} \right\rangle$ and

$$\begin{aligned} D_{\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle} f(1,1) &= \left\langle \frac{1}{\sqrt{2}} + e, \frac{1}{\sqrt{2}} \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \\ &= \frac{1}{2} + \frac{e}{\sqrt{2}} + \frac{1}{2} = 1 + \frac{e}{\sqrt{2}}. \end{aligned}$$

(2) You are driving your car towards the northeast at 72km/h along a terrain whose elevation at the point (x,y) is $\frac{1}{8+x^2+y^2}$ (all distances are measured in kilometres). What is your rate of ascent/descent when your car is at the location $(1,1)$? What about if the location was $(1,-1)$?

Solution 1: Let $f(x,y) = \frac{1}{8+x^2+y^2}$ and Let $\vec{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, a unit vector in the northeast direction. Then the slope in the direction of travel is:

$$\left(\vec{\nabla} f\right)(x,y) \cdot \vec{u} = \left\langle \frac{-2x}{(8+x^2+y^2)^2}, \frac{-2y}{(8+x^2+y^2)^2} \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

which at $(1,1)$ is

$$-\frac{1}{50\sqrt{2}} \langle 1,1 \rangle \cdot \langle 1,1 \rangle = -\frac{1}{25\sqrt{2}}.$$

In other words, for every unit of horizontal distance in the direction of travel one would lose $\frac{1}{25\sqrt{2}}$ units of elevation. We are travelling at 72km/h and are therefore losing altitude at the rate of $\frac{72}{25\sqrt{2}}$ kilometers per hour, or $\frac{2\sqrt{2}}{5} \frac{\text{m}}{\text{s}}$. If $(x,y) = (-1,1)$ then the directional derivative would have been $-\frac{1}{50\sqrt{2}} \langle 1,-1 \rangle \cdot \langle 1,1 \rangle = 0$, so the the car is momentarily level.

Solution 2: Let $\vec{v} = 72 \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ be the velocity vector. Then the rate of change is $\left(\vec{\nabla} f\right)(x,y) \cdot \vec{v} = -\frac{72}{25\sqrt{2}}$. Similarly, $-\frac{1}{50} \langle 1,-1 \rangle \cdot 72 \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = 0$.

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(3) An ant is crawling along the curve $y = x^2$ at the rate of $v$cm/s (distances are measured in cm). The temperature in the xy plane is varying according to $T(x, y) = \frac{y}{1+x^2}$. What is the rate of change of the temperature the ant sees when it is located at (x, y)?

Solution 2: A tangent vector to the curve $y = x^2$ at (x, y) is given by $\langle 1, 2x \rangle$. A *unit* tangent vector is therefore $\vec{u} = \frac{1}{1+4x^2} \langle 1, 2x \rangle$. The temperature gradient is

$$\vec{\nabla} T(x, y) = \left\langle \frac{-2xy}{(1 + x^2)^2}, \frac{1}{1 + x^2} \right\rangle$$

so, per unit distance travelled, the ant sees a temperature change of

$$\vec{\nabla} T \cdot \vec{u} = \frac{-2xy + (1 + x^2)2x}{(1 + x^2)^2(1 + 4x^2)} = \frac{2x(1 + x^2 - y)}{(1 + x^2)^2(1 + 4x^2)} .$$

Now the ant is travelling at the rate of v units of distance per unit time, so per unit time the temperature change is

$$v\vec{\nabla} T \cdot \vec{u} = \frac{2x(1 + x^2 - y)}{(1 + x^2)^2(1 + 4x^2)} v .$$