MATH 253 – WORKSHEET 15
DIRECTIONAL DERIVATIVES

(1) An ant is crawling along the curve $y = x^2$ at the rate of $v$cm/s (distances are measured in cm). The temperature in the xy plane is varying according to $T(x, y) = \frac{y}{1+x^2}$. What is the rate of change of the temperature the ant sees when it is located at (x, y)?

Solution: At (x_0, y_0) the tangent line to $y = x^2$ has slope $2x_0$ (the derivative $\frac{dy}{dx}$), and so is parallel to the vector$\langle 1, 2x_0\rangle$ [for a detailed derivation, the tangent line: has the equation $y = y_0 + 2x_0 (x - x_0) = 2x_0 x - x_0^2$ and therefore the parametrization $\left(x, 2x_0 x - x_0^2\right) = \left(0, -x_0^2\right) + x \langle 1, 2x_0\rangle$]. A unit vector in the direction of travel is therefore $\vec{u} = \frac{1}{\sqrt{1+4x_0^2}} \langle 1, 2x_0\rangle$. The gradient of the temperature is

$$\begin{aligned}\vec{\nabla} T (x_0, y_0) &= \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y} \right\rangle \\ &= \left\langle \frac{-2x_0 y_0}{(1 + x_0^2)^2}, \frac{1}{1 + x_0^2} \right\rangle \\ &= \frac{1}{(1 + x_0^2)^2} \left\langle -2x_0 y_0, 1 + x_0^2 \right\rangle .\end{aligned}$$

The directional derivative in the direction $\vec{u}$ (measuring rate of change of tempreature per unit distance travelled) is then

$$D_{\vec{u}} T = \frac{1}{\sqrt{1 + 4x_0^2}\,(1 + x_0^2)^2} \left\langle -2x_0 y_0, 1 + x_0^2 \right\rangle \cdot \langle 1, 2x_0\rangle ,$$

and the rate of change (measuring change per unit time) is thus

$$v D_{\vec{u}} T = \frac{2x_0^3 + 2x_0 - 2x_0 y_0}{\sqrt{1 + 4x_0^2}\,(1 + x_0^2)^2} v .$$

(2) Show that every plane tangent to the surface $z^2 = x^2 + y^2$ passes through the origin.

Solution: Let $F(x, y, z) = x^2 + y^2 - z^2$. Then the gradient vector at any point is tangent to the level surface at that point. Since $\vec{\nabla} F (x_0, y_0, z_0) = \langle 2x_0, 2y_0, -2z_0\rangle$, the equation of the plane tangent at (x_0, y_0, z_0) where $F (x_0, y_0, z_0) = 0$ is therefore

$$2x_0 (x - x_0) + 2y_0 (y - y_0) - 2z_0 (z - z_0) = 0 ,$$

or

$$x_0 x + y_0 y - z_0 z = z_0^2 - x_0^2 - y_0^2 = 0 .$$

Clearly $(0, 0, 0)$ solves this equation.

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