MATH 253 – WORKSHEET 16
OPTIMIZATION

1. CRITICAL POINTS

1.1. Single-variable.

Definition 1. $f(x)$ has a *critical point* at x_0 if $f'(x_0) = 0$. If, in addition, $f''(x_0) \neq 0$ call the point *"ordinary"*, and (fact) if $f''(x_0) > 0$ we have a *local minimum*, if $f''(x_0) < 0$ a *local maximum*.

Given $f(x)$ defined on $[a, b]$ we find absolute minimum/maximum by (1) Finding the critical points in (a, b); (2) Evaluating f at every critical point *and at the endpoints* a, b; and (3) Selecting the smallest/largest value seen.

1.2. Two-variable.

Definition 2. $f(x, y)$ has a *critical point* at (x_0, y_0) if $\vec{\nabla} f\,(x_0, y_0) = 0$. In that case set $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$ (evaluated at (x_0, y_0)). If $D \neq 0$ the point *"ordinary"*, and further:

- If $D < 0$ we have a *saddle point*
- If $D > 0$, then $f_{xx} > 0$ at a *local minimum*, $f_{xx} < 0$ at a *local maximum*.

Minimum-finding: given $f(x, y)$ defined on a region R, (1) find the critical points inside R (2) evaluate f on the boundary of R (3) select the smallest/largest value.

2. PROBLEMS

(1) Let $f(x, y) = (2x - x^2)(2y - y^2)$.

(a) Find and classify the critical points

Solution: $\vec{\nabla} f = \left\langle (2 - 2x)(2y - y^2), (2x - x^2)(2 - 2y \right\rangle = 2\left\langle (1 - x)y(2 - y), (2 - x)x(1 - y) \right\rangle$. Thus $\frac{\partial f}{\partial x} = 0$ if $x = 1$ or $y = 0$ or $y = 2$. If $x = 1$ then $\frac{\partial f}{\partial y} = 2(1 - y)$ so we have a critical point when $y = 1$. If $y = 0$ or 2 then $\frac{\partial f}{\partial y} = \pm 2(2 - x)x$ and in either case we get a critical point of $x = 0$ or $x = 2$. To conclude, the critical points are $(1, 1)$,$(0, 0)$, $(0, 2)$, $(2, 0)$, $(2, 2)$.

To classify them we calculate the second derivatives $f_{xx} = -2y(2 - y)$, $f_{yy} = -2x(2 - x)$ and $f_{xy} = 4(1 - x)(1 - y)$. Thus $D = f_{xx}f_{yy} - f_{xy}^2 = 4xy(2 - x)(2 - y) - 16(1 - x)^2(1 - y)^2$ and we have

Point	D	f_{xx}	type
$(1, 1)$	4	-2	local max
$(0, 0)$	-16		saddle point
$(0, 2)$	-16		saddle point
$(2, 0)$	-16		saddle point
$(2, 2)$	-16		saddle point

(b) Find the absolute maximum and minimum in the domain $R = \{0 \leq x \leq 2,\ 0 \leq y \leq 2\} = [0, 2] \times [0, 2]$.

Solution: The only critical point in the domain is $(1, 1)$ and $f\,(1, 1) = 1$. On the boundary we have either $x = 0$ or $x = 2$ or $y = 0$ or $y = 2$ and in any case $f(x, y) = 0$. Thus the maximum is 1 and occurs at $(1, 1)$, while the minimum is 0 and it occurs on the boundary.

(c) Find the absolute maximum and minimum in the domain $R = \{0 \leq x \leq 3,\ 0 \leq y \leq 2\} = [0, 3] \times [0, 2]$.

Solution: Again the only critical point inside the domain is $(1, 1)$where $f\,(1, 1) = 1$. On the boundary, if $x = 0$ or $y = 0$ or $y = 2$ then $f(x, y) = 0$ but when $x = 3$ and $0 \leq y \leq 2$ we

Date: 16/10/2013.

have $f(3,y) = -3y(2-y) = 3y^2 - 6y$ which is non-positive on $[0,2]$ (both y and $2-y$ are non-negative there). In particular, $f \le 0$ on the boundary and the maximum is 1 at $(1,1)$ like before. Since $f(3,y) = 3\left((y-1)^2 - 1\right)$ we see that the minimum on the boundary is -3, occuring at $(3,1)$ and this is the absolute minimum.

(2) Find the equation of the plane which passes through $(1,2,3)$ and encloses the smallest volume in the positive octant.

Solution: Suppose the plane meets the x, y, z axes at a, b, c respectively ("parametrization" / "naming of variables" – we parametrize the plane by its axis intercepts, and will call them a, b, c). Then the volume of the resulting pyramid is $V = \frac{1}{2}abc$ (it is $\frac{1}{2}\left(\langle a,0,0\rangle \times \langle 0,b,0\rangle\right) \cdot (0,0,c)$). Suppose the equation of the plane was $Ax + By + Cz = 1$. Plugging in the three points we see that $A = \frac{1}{a}$ and so on, so the equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. That the plane passes through $(1,2,3)$ then means $\frac{1}{a} + \frac{2}{b} + \frac{3}{c} = 1$. Then $\frac{1}{a} = 1 - \frac{2}{b} - \frac{3}{c} = \frac{bc-2c-3b}{bc}$ and hence

$$a = \frac{bc}{bc - 2c - 3b}\,.$$

We thus have $V(b,c) = \frac{1}{2}\frac{b^2c^2}{bc-2c-3b}$. We look for critical points:

$$\frac{\partial V}{\partial b} = \frac{2bc^2(bc-2c-3b) - b^2c^2(c-3)}{2\left(bc-2c-3b\right)^2} = \frac{bc^2\left(bc-4c-3b\right)}{\left(bc-2c-3b\right)^2}$$

and

$$\frac{\partial V}{\partial c} = \frac{2b^2c(bc-2c-3b) - b^2c^2(b-2)}{2\left(bc-2c-3b\right)^2} = \frac{b^2c(bc-2c-6b)}{2\left(bc-2c-3b\right)^2}\,.$$

Since $b, c \neq 0$ (the plane can't pass through the origin) this vanishes when

$$\begin{cases} bc - 4c - 3b &= 0 \\ bc - 2c - 6b &= 0 \end{cases}$$

Subtracting the equations we find $-2c + 3b = 0$, so $c = \frac{3}{2}b$. Plugging back in we get $\frac{3}{2}b^2 - 6b - 3b = 0$ and since $b \neq 0$ we get $b = 6$, hence $c = 9$ and $a = 3$ so the plane is $\frac{x}{3} + \frac{y}{6} + \frac{z}{9} = 1$ with the volume $\frac{1}{2}3 \cdot 6 \cdot 9 = 81$.