MATH 253 – WORKSHEET 22
ITERATED INTEGRALS ON PLANAR DOMAINS

(1) Let D be the finite region bounded by the curves $x = y$ and $x = 2 - y^2$. Find $\iint_D y \, \mathrm{d}A$, slicing the domain vertically.

Solution:

(2) Let $D = \left\{x^2 + y^2 \leq 4\right\}$. Evaluate $\iint_R \left(e^y x^2 \tan\left(\frac{x}{2}\right) + \sin(y^3) + 5\right) \mathrm{d}A$.

Solution: $\iint_D \left(e^y x^2 \tan\left(\frac{x}{2}\right) + \sin(y^3) + 5\right) \mathrm{d}A = \iint_D e^y x^2 \tan\left(\frac{x}{2}\right) \mathrm{d}A + \iint_D \sin\left(y^3\right) \mathrm{d}A + \iint_D 5 \, \mathrm{d}A$. Now the first summand is odd in x, and the domain is symmetric under reflection in the y-axis, so $\iint_D e^y x^2 \tan\left(\frac{x}{2}\right) \mathrm{d}A = 0$. Similarly, $\sin\left(y^3\right)$ is odd in y and the domain is symmetric under reflection in the x-axis, so $\iint_D \tan\left(y^3\right) \mathrm{d}A = 0$. Finally, $\iint_D 5 \, \mathrm{d}A = 5 \iint_D 1 \, \mathrm{d}A = 5 \, \mathrm{Area}(D) = 5\pi \cdot 2^2 = 20\pi$.

(3) Integrate $f(x, y) = e^{y^2}$ on the triangle with vertices $(0, 0)$, $(0, 3)$, $(1, 3)$.

Solution: Slicing vertically, x ranges in $[0, 1]$ and for each x we have $3x \leq y \leq 3$ ($y = 3x$ is the equation of the line connecting $(0, 0)$ to $(1, 3)$). The integral is therefore

$$\int_{x=0}^{x=1} \mathrm{d}x \int_{y=3x}^{y=3} \mathrm{d}y e^{y^2}$$

<u>**OOPS**</u>: we don't know an antiderivative for e^{y^2}, so we try slicing horizontally instead. Now the integral is

$$\int_{y=0}^{y=3} \mathrm{d}y \int_{x=0}^{x=y/3} \mathrm{d}x e^{y^2} = \int_{y=0}^{y=3} e^{y^2} \frac{y}{3} \, \mathrm{d}y = \frac{1}{6} \left[e^{y^2}\right]_{y=0}^{y=3} = \frac{e^9 - 1}{6} .$$

(4) Reverse the order of integration in $\int_{x=1}^{x=2} \int_{y=0}^{\ln x} f(x, y) \, \mathrm{d}y \, \mathrm{d}x$.

Solution: The range of y values is between $y = 0$ and $y = \ln 2$ (the largest upper bound on y). Given y, we see that (x, y) is in the region if $1 \leq x \leq 2$ (from bounds on the first integral) and also $y \leq \ln x$ (bound on the second integral). The latter condition can be written as $x \geq e^y$, so we must have $1 \leq x \leq 2$ and also $x \geq e^y$. Now in our region $y \geq 0$ so $e^y \geq e^0 = 1$ so the condition $x \geq 1$ is redundant. Also, if $y \leq \ln 2$ then $e^y \leq 2$ so the interval $[e^y, 2]$ is always non-empty. We conlcude that the integral is also

$$\int_{y=0}^{y=\ln 2} \int_{x=e^y}^{x=2} f(x, y) \, \mathrm{d}x \, \mathrm{d}y .$$

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