MATH 253 – WORKSHEET 23
POLAR COORDINATES AND INTEGRATION

1. Polar coordinates

(1) Let $D = \{(x,y) \mid 1 \le x^2 + y^2 \le 2,\ x, y \ge 0\}$.

(a) Express D in the form $D = \{(r,\theta) \mid ??\}$

Solution: The first condition reads $1 \le r^2 \le 2$ or $1 \le r \le \sqrt{2}$. For the second condition, geometrically we see that $0 \le \theta \le \frac{\pi}{2}$. Algebraically, $x, y \ge 0$ means both $r\cos\theta \ge 0$ and $r\sin\theta \ge 0$. Since r is non-negative this means $\sin\theta, \cos\theta \ge 0$. The first means $0 \le \theta \le \pi$, and the second means $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so we again get $0 \le \theta \le \frac{\pi}{2}$ and the region is $\{(r,\theta) \mid 1 \le r \le \sqrt{2},\ 0 \le \theta \le \frac{\pi}{2}\}$.

(b) Try expressing $\iint_D \cos(x^2+y^2)\,\mathrm{d}A$ as an iterated integral, slicing the domain vertically.

Solution: For $0 \le x \le 1$, the vertical slices begin at the inner circle and end on the outer circle. For $1 \le x \le \sqrt{2}$, the vertical slices begin at the x-axis and end on the outer circle. The integral is therfore

$$\int_{x=0}^{x=1} \mathrm{d}x \int_{y=\sqrt{1-x^2}}^{y=\sqrt{2-x^2}} \mathrm{d}y \cos(x^2+y^2) + \int_{x=1}^{x=\sqrt{2}} \mathrm{d}x \int_{y=0}^{y=\sqrt{2-x^2}} \mathrm{d}y \cos(x^2+y^2)$$

(c) Calculate $\iint_D \cos(x^2+y^2)\,\mathrm{d}A$ in polar coordinates.

Solution: This is

$$\begin{aligned}\int_{\theta=0}^{\theta=\frac{\pi}{2}} \mathrm{d}\theta \int_{r=1}^{r=\sqrt{2}} r\,\mathrm{d}r \cos(r^2) &= \left(\int_{\theta=0}^{\theta=\frac{\pi}{2}} \mathrm{d}\theta\right)\left(\int_{r=1}^{r=\sqrt{2}} \cos(r^2)\, r\,\mathrm{d}r\right) \\ &= \frac{\pi}{2}\left[\frac{1}{2}\sin(r^2)\right]_{r=1}^{r=\sqrt{2}} = \frac{\pi(\sin 2 - \sin 1)}{4}.\end{aligned}$$

(2) Find the volume of the solid lying above the xy-plane, below the paraboloid $z = x^2 + y^2$ and inside the cylinder $(x-1)^2 + y^2 = 1$.

(a) Find a region R in the plane and a function $f(x,y)$ so that the volume is $\iint_R f(x,y)\,\mathrm{d}A$.

Solution: Points inside the cylnder have (x,y) belonging to $R = \{(x,y) \mid (x-1)^2 + y^2 \le 1\}$. Volume is then

$$\iint_R z\,\mathrm{d}A = \iint_R (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$$

(b) Write R and f in polar coordinates.

Solution: Setting $x = r\cos\theta$, $y = r\sin\theta$, we have $r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta \le 1$. Subtracting the 1 and using $\cos^2\theta + \sin^2\theta = 1$ this isequivalent to $r^2 \le 2r\cos\theta$, and dividing by r (which is always positive) this is equivalent to $r \le 2\cos\theta$. Since the region is on the right of the y axis, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so $R = \{(r,\theta) \mid -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ r \le 2\cos\theta\}$. $x^2 + y^2 = r^2$ so $f(r,\theta) = r^2$.

(c) Evaluate the integral.

Solution: See solution to WS 24

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