MATH 253 – WORKSHEET 25
MASS AND CENTER OF MASS

Find the center of mass of the region inside $x^2+y^2=2y$ and outside $x^2+y^2=1$ if the density is inversely proportional to the distance from the origin.

Solution: Note that $x^2+y^2=2y \iff x^2+(y^2-2y+1)=1 \iff x^2+(y-1)^2=1$ which is also a circle, so the region is

$$R=\left\{(x,y)\mid x^2+y^2\geq 1,\, x^2+(y-1)^2\leq 1\right\}=\left\{(x,y)\mid x^2+y^2\geq 1,\, x^2+y^2\leq 2y\right\}\ .$$

(1) We first find the total mass. The density is proprtional to $\frac{1}{\sqrt{x^2+y^2}}$ so the mass is

$$\iint_R \frac{\mathrm{d}A}{\sqrt{x^2+y^2}}\ .$$

Tempting to switch to polar coordinates. Then the region R is given by $R=\left\{(r,\theta)\mid r^2\geq 1,\, r^2\leq 2r\sin\theta\right\}$, that is $\{(r,\theta)\mid 1\leq r\leq 2\sin\theta\}$. Note that $2\sin\theta\geq 1$ only if $\sin\theta\geq\frac{1}{2}$, only if $\frac{\pi}{6}\leq\theta\leq\pi-\frac{\pi}{6}$, so the region can be written as

$$\left\{(r,\theta)\mid \frac{\pi}{6}\leq\theta\leq\frac{5\pi}{6},\, 1\leq r\leq 2\sin\theta\right\}\ .$$

We therefore get the integral

$$\begin{aligned}
M &= \int_{\theta=\frac{\pi}{6}}^{\theta=\frac{5\pi}{6}}\mathrm{d}\theta\int_{r=1}^{r=2\sin\theta}\frac{r\,\mathrm{d}r}{r}\\
&= \int_{\theta=\frac{\pi}{6}}^{\theta=\frac{5\pi}{6}}\mathrm{d}\theta\,[2\sin\theta-1]\\
&= [-2\cos\theta-\theta]_{\theta=\frac{\pi}{6}}^{\theta=\frac{5\pi}{6}}\\
&= -2\cos\frac{5\pi}{6}-\frac{5\pi}{6}+2\cos\frac{\pi}{6}+\frac{\pi}{6}=4\cos\frac{\pi}{6}-\frac{4\pi}{6}\\
&= 4\frac{\sqrt{3}}{2}-\frac{4\pi}{6}=2\sqrt{3}-\frac{2}{3}\pi\,.
\end{aligned}$$

(2) We next find $\bar{x}$. Since the region and the density are both symmetric about the y-axis the average x coordinate is $\bar{x}=0$.

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(3) Finally,

$$
\begin{aligned}
\bar{y} &= \frac{\iint_R y\rho(x,y)\,\mathrm{d}A}{M} = \frac{1}{M}\int_{\theta=\frac{\pi}{6}}^{\theta=\frac{5\pi}{6}} \mathrm{d}\theta \int_{r=1}^{r=2\sin\theta} (r\sin\theta)\left(\frac{1}{r}\right) r\,\mathrm{d}r \\
&= \frac{1}{M}\int_{\theta=\frac{\pi}{6}}^{\theta=\frac{5\pi}{6}} \mathrm{d}\theta \sin\theta \int_{r=1}^{r=2\sin\theta} r\,\mathrm{d}r \\
&= \frac{1}{M}\int_{\theta=\frac{\pi}{6}}^{\theta=\frac{5\pi}{6}} \mathrm{d}\theta \sin\theta \left[\frac{r^2}{2}\right]_{r=1}^{r=2\sin\theta} \\
&= \frac{1}{M}\int_{\theta=\frac{\pi}{6}}^{\theta=\frac{5\pi}{6}} \mathrm{d}\theta \sin\theta \left[2\sin^2\theta - \frac{1}{2}\right] \\
&\overset{u=\cos\theta}{=} \frac{1}{M}\int_{\sqrt{3}/2}^{-\sqrt{3}/2} (-\,\mathrm{d}u)\left[2(1-u^2) - \frac{1}{2}\right] \\
&= \frac{2}{M}\int_0^{\sqrt{3}/2} \left[\frac{3}{2} - 2u^2\right] \mathrm{d}u = \frac{1}{M}\left[3u - \frac{4}{3}u^3\right]_0^{\sqrt{3}/2} \\
&= \frac{1}{M}\left[\frac{3\sqrt{3}}{2} - \frac{4}{3}\frac{3\sqrt{3}}{8}\right] \\
&= \frac{\sqrt{3}}{2\sqrt{3} - \frac{2}{3}\pi} = \frac{1}{2 - \frac{2\pi}{3\sqrt{3}}}\,.
\end{aligned}
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