MATH 253 – WORKSHEET 28
TRIPLE INTEGRALS

(1) Integrate $f(x,y,z) = xy^2z^3$ on the box $[5,6]\times[3,4]\times[1,2]$.
Solution: We convert to an iterated integral, and notice that it factors:

$$\begin{aligned}
\iiint\limits_{[5,6]\times[3,4]\times[1,2]} xy^2z^3\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_{x=5}^{x=6}\mathrm{d}x\int_{y=3}^{y=4}\mathrm{d}y\int_{z=1}^{z=2}\mathrm{d}z xy^2z^3 \\
&= \int_{x=5}^{x=6}\mathrm{d}x x\int_{y=3}^{y=4}\mathrm{d}y y^2\int_{z=1}^{z=2}\mathrm{d}z z^3 \\
&= \left(\int_{x=5}^{x=6}\mathrm{d}x x\right)\left(\int_{y=3}^{y=4}\mathrm{d}y y^2\right)\left(\int_{z=1}^{z=2}\mathrm{d}z z^3\right) \\
&= \left[\frac{x^2}{2}\right]_{x=5}^{x=6}\cdot\left[\frac{y^3}{3}\right]_{y=3}^{y=4}\cdot\left[\frac{z^4}{4}\right]_{z=1}^{z=2} \\
&= \frac{11}{2}\cdot\frac{37}{3}\cdot\frac{15}{4} = \frac{2035}{8}\,.
\end{aligned}$$

Important for the factorization: not only did the *integrand* factorize $(x\cdot y^2\cdot z^3)$ but the *domain* also factors (that is, it's a box).

(2) Calculate $\iiint_B xye^{xz}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ where $B = [1,2]^3$.
Solution: We convert to an iterated integral.

$$\begin{aligned}
\iiint_B xye^{xz}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_{x=1}^{x=2}\mathrm{d}x\int_{y=1}^{y=2}\mathrm{d}y\int_{z=1}^{z=2}\mathrm{d}z xye^{xz} \\
&\overset{u=xz}{=} \int_{x=1}^{x=2}\mathrm{d}x\int_{y=1}^{y=2}\mathrm{d}y y\int_{u=x}^{u=2x}\mathrm{d}u e^u \\
&= \int_{x=1}^{x=2}\mathrm{d}x\int_{y=1}^{y=2}\mathrm{d}y y\left[e^{2x}-e^x\right] \\
&= \left(\int_{x=1}^{x=2}\left(e^{2x}-e^x\right)\mathrm{d}x\right)\left(\int_{y=1}^{y=2}\mathrm{d}y y\right) \\
&= \left[\frac{1}{2}e^{2x}-e^x\right]_{x=1}^{x=2}\cdot\left[\frac{y^2}{2}\right]_{y=1}^{y=2} \\
&= \left[\frac{1}{2}e^4-e^2-\frac{1}{2}e^2+e^1\right]\left[\frac{4-1}{2}\right] \\
&= \frac{3}{4}\left[e^4-3e^2+2e\right]\,.
\end{aligned}$$

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