MATH 253 – WORKSHEET 30
TRIPLE INTEGRALS AND APPLICATIONS

(1) Consider the iterated integral $\int_{x=0}^{x=1} \mathrm{d}x \int_{y=\sqrt{x}}^{y=1} \mathrm{d}y \int_{z=0}^{z=1-y} \mathrm{d}z f$. Write the other 5 equivalent integrals coming from changing the order of integration.

Solution: See WS 29.

(2) Find the volume and the center-of-mass of the solid bounded by the parabolic cylinder $y = x^2$, the xy plane, and the plane $y + z = 1$.

Solution: The plane $y + z = 1$ intersects the xy plane (where $z = 0$) in the line $y = 1$. Let R be the region in the plane bounded by the parabola $y = x^2$ and the line $y = 1$. The solid then consists of the points above R and below the plane $y + z = 1$ [why? either draw a picture (which is enough for this course) or compare with the solid bound by the cylinder, the xy plane, and the plane $y = 1$ which contains the original solid (since the plane $y = 1$ is always "farther out" than $y + z = 1$ if $z \geq 0$) and by construction has base R]. Considering a point (x, y) in R, the set of points (x, y, z) in our solid lying above it is a "z-line" beginning at the base (xy plane, $z = 0$) and ending at the "roof" plane $y + z = 1$. Converting the two endpoints to statements about z, integrals over the solid will be of the form $\iint_R \mathrm{d}x\,\mathrm{d}y \int_{z=0}^{z=1-y} \mathrm{d}z f(x, y, z)$. We can slice the region R itself horizontally or vertically.

- Slicing vertically, the x range is $[-1, 1]$. In this range every verticle "y line" begins at at the parabolic $y = x^2$ and ends at the line $y = 1$. The full integral is then
$$\int_{x=-1}^{x=1} \mathrm{d}x \int_{y=x^2}^{y=1} \mathrm{d}y \int_{z=0}^{z=1-y} \mathrm{d}z f(x, y, z)\,.$$
- Slicing horizontally, the y range is $[0, 1]$. In this range every horizontal "x line" beings at the left arm of the parabola (at a point where $y = x^2$ so $x = -\sqrt{y}$) and ends at the right arm (where $x = \sqrt{y}$). Both endpoints satisfy $y = x^2$ but are not the same point, since y has *two* square roots. The full integral is also
$$\int_{y=0}^{y=1} \mathrm{d}y \int_{x=-\sqrt{y}}^{x=\sqrt{y}} \mathrm{d}x \int_{z=0}^{z=1-y} \mathrm{d}z f(x, y, z)\,.$$

  Remark: Note that the equalities $y = x^2$ only hold at the *endpoints* of our "x line", not *inside the domain.*

Volume: The volume V of the region is

$$\begin{aligned}
\int_{x=-1}^{x=1} \mathrm{d}x \int_{y=x^2}^{y=1} \mathrm{d}y \int_{z=0}^{z=1-y} \mathrm{d}z \cdot 1 &= \int_{x=-1}^{x=1} \mathrm{d}x \int_{y=x^2}^{y=1} \mathrm{d}y (1-y) \\
&= \int_{x=-1}^{x=1} \mathrm{d}x \left[y - \frac{y^2}{2}\right]_{y=x^2}^{y=1} \\
&= \int_{x=-1}^{x=1} \mathrm{d}x \left(1 - \frac{1}{2} - x^2 + \frac{x^4}{2}\right) \\
&= \left[\frac{1}{2}x - \frac{x^3}{3} + \frac{x^5}{10}\right]_{x=-1}^{x=1} \\
&= 2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{10}\right) = \frac{2(15-10+3)}{30} = \frac{8}{15}\,.
\end{aligned}$$

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Or equivalently

$$\begin{aligned}
\int_{y=0}^{y=1} \mathrm{d}y \int_{x=-\sqrt{y}}^{x=\sqrt{y}} \mathrm{d}x \int_{z=0}^{z=1-y} \mathrm{d}z \cdot 1 &= \int_{y=0}^{y=1} \mathrm{d}y \int_{x=-\sqrt{y}}^{x=\sqrt{y}} \mathrm{d}x\,(1-y) \\
&= \int_{y=0}^{y=1} \mathrm{d}y(1-y)2\sqrt{y} \\
&= 2\left[\frac{2}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_{y=0}^{y=1} \\
&= 4\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{8}{15}\,.
\end{aligned}$$

Center of mass: The region is symmetric under reflection in the x-variable. This is clear from the definitions (which only involve x^2), or from the integral (where the bounds on the x integral are symmetric and any later dependence is on x^2), so $\bar{x} = 0$. We will need to integrate to find $\bar{y}$, $\bar{z}$. Using the first form of the integral, we have

$$\begin{aligned}
\bar{y} &= \frac{1}{\text{volume}} \int_{x=-1}^{x=1} \mathrm{d}x \int_{y=x^2}^{y=1} \mathrm{d}y \int_{z=0}^{z=1-y} \mathrm{d}z \cdot y \\
&= \frac{1}{8/15} \int_{x=-1}^{x=1} \mathrm{d}x \int_{y=x^2}^{y=1} \mathrm{d}y(1-y)y \\
&= \frac{15}{8} \int_{x=-1}^{x=1} \mathrm{d}x \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_{y=x^2}^{y=1} \\
&= \frac{15}{8} \int_{x=-1}^{x=1} \mathrm{d}x \left[\frac{1}{2} - \frac{1}{3} - \frac{x^4}{2} + \frac{x^6}{3}\right] \\
&= \frac{15}{8} \left[\frac{x}{6} - \frac{x^5}{10} + \frac{x^7}{21}\right]_{x=-1}^{x=1} \\
&= \frac{15}{8} 2 \left[\frac{1}{6} - \frac{1}{10} + \frac{1}{21}\right] = \frac{15}{4}\left[\frac{35-21+10}{210}\right] = \frac{3}{7}
\end{aligned}$$

and

$$\begin{aligned}
\bar{z} &= \frac{15}{8} \int_{x=-1}^{x=1} \mathrm{d}x \int_{y=x^2}^{y=1} \mathrm{d}y \int_{z=0}^{z=1-y} \mathrm{d}z \cdot z \\
&= \frac{15}{8} \int_{x=-1}^{x=1} \mathrm{d}x \int_{y=x^2}^{y=1} \mathrm{d}y \frac{(1-y)^2}{2} \\
&= \frac{15}{8} \int_{x=-1}^{x=1} \mathrm{d}x \left[\frac{(y-1)^3}{6}\right]_{y=x^2}^{y=1} \\
&= \frac{15}{8} \int_{x=-1}^{x=1} \mathrm{d}x \left[-\frac{(x^2-1)^3}{6}\right] \\
&= \frac{15}{8 \cdot 6} \cdot 2 \int_{x=0}^{x=1} \mathrm{d}x \left(1 - 3x^2 + 3x^4 - x^3\right) \\
&= \frac{5}{8} \left[x - x^3 + \frac{3x^5}{5} - \frac{x^7}{7}\right]_{x=0}^{x=1} \\
&= \frac{5}{8} \left[1 - 1 + \frac{3}{5} - \frac{1}{7}\right] = \frac{5}{8}\left[\frac{21-5}{35}\right] = \frac{2}{7}\,.
\end{aligned}$$

In other other order of integration the integrals would look like

$$\begin{aligned}
\bar{y} &= \frac{15}{8}\int_{y=0}^{y=1} \mathrm{d}y \int_{x=-\sqrt{y}}^{x=\sqrt{y}} \mathrm{d}x \int_{z=0}^{z=1-y} \mathrm{d}z \cdot y \\
&= \frac{15}{8}\int_{y=0}^{y=1} \mathrm{d}y \int_{x=-\sqrt{y}}^{x=\sqrt{y}} \mathrm{d}x y(1-y) \\
&= \frac{15}{8}\int_{y=0}^{y=1} \mathrm{d}x 2\sqrt{y} \cdot y(1-y) \\
&= \frac{15}{4}\int_{y=0}^{y=1} \left(y^{3/2} - y^{5/2}\right) \\
&= \frac{15}{4}\left(\frac{2}{5} - \frac{2}{7}\right) = \frac{15}{2}\left(\frac{7-5}{35}\right) = \frac{3}{7}
\end{aligned}$$

and

$$\begin{aligned}
\bar{z} &= \frac{15}{8}\int_{y=0}^{y=1} \mathrm{d}y \int_{x=-\sqrt{y}}^{x=\sqrt{y}} \mathrm{d}x \int_{z=0}^{z=1-y} \mathrm{d}z \cdot z \\
&= \frac{15}{8}\int_{y=0}^{y=1} \mathrm{d}y \int_{x=-\sqrt{y}}^{x=\sqrt{y}} \mathrm{d}x \frac{(1-y)^2}{2} \\
&= \frac{15}{8}\int_{y=0}^{y=1} \mathrm{d}y 2\sqrt{y} \frac{(1-y)^2}{2} \\
&= \frac{15}{8}\int_{y=0}^{y=1} \left(y^{5/2} - 2y^{3/2} + y^{1/2}\right) \mathrm{d}y \\
&= \frac{15}{8}\left[\frac{2}{7} - 2\cdot\frac{2}{5} + \frac{2}{3}\right] = \frac{15}{4}\cdot\frac{15-42+35}{105} = \frac{2}{7}
\end{aligned}$$