MATH 253 – WORKSHEET 31
CYLINDRICAL COORDINATES

(1) Express the following surfaces in cylindrical coordinates.
   (a) The cylinder of radius 2 about the z-axis.
   Solution: $r = 2$.
   (b) The paraboloid $z = x^2 + y^2$.
   Solution: $z = r^2$.

(2) A drill bit of diameter a is used to drill a hole through a ball of radius a. What is the volume of the remaining object?

Solution: We work in cylindrical coordinates, with the z-axis being in the middle of the drilled cylinder – (1) the problem has an axis of symmetry, and (2) we want to "walk" up and down the axis. We first find the equations of the bounding surfaces: the sphere around the ball has the equation $x^2 + y^2 + z^2 = a^2$, so in our coordinates $r^2 + z^2 = a^2$. The cylinder in the middle has the equation $r = \frac{a}{2}$ (note we are given the diameter!).

- Slicing method 1: Projecting from the top to the xy plane we get the an annula "shadow", specifically $\frac{a}{2} \leq r \leq a$. Above each point (r, θ) in the xy plane we see a "z-line" extending from the bottom hemisphere to the top hemisphere, that is from $z = -\sqrt{a^2 - r^2}$ to $z = +\sqrt{a^2 - r^2}$. The volume is then

$$
\begin{aligned}
V &= \int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{r=a/2}^{r=a} r\,\mathrm{d}r \int_{z=-\sqrt{a^2-r^2}}^{z=+\sqrt{a^2-r^2}} \mathrm{d}z \cdot 1 \\
&= (2\pi) \int_{r=a/2}^{r=a} r\,\mathrm{d}r 2\sqrt{a^2 - r^2} \\
&\overset{u=a^2-r^2}{=} (2\pi) \int_{u=\frac{3}{4}a^2}^{u=0} (-\,\mathrm{d}u)\sqrt{u} \\
&= (2\pi) \int_{u=0}^{u=\frac{3}{4}a^2} \sqrt{u}\,\mathrm{d}u \\
&= (2\pi)\left(\frac{2}{3}\left(\frac{3}{4}a^2\right)^{3/2}\right) = \frac{\sqrt{3}\pi}{2}a^3\,.
\end{aligned}
$$

- Slicing method 2: Slice in planes parallel to the xy plane (that is, planes of fixed z). For each z, θ, an "r-line" is a ray extending perpendicular to the z-axis. It will begin at the cylinder and end at the sphere, so the range for the r-integral will be $\frac{a}{2} \leq r \leq \sqrt{z^2 - r^2}$. The z-range begins and ends at the meeting points of the cylinder and the sphere, that is where $r = \frac{a}{2}$ *and*

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$r^2 + z^2 = a^2$ that is where $z = \pm\frac{\sqrt{3}}{2}a$. The volume is thus also

$$\begin{aligned}
V &= \int_{z=-\frac{\sqrt{3}}{2}a}^{z=+\frac{\sqrt{3}}{2}a} \mathrm{d}z \int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{r=\frac{a}{2}}^{r=\sqrt{a^2-z^2}} r\,\mathrm{d}r \\
&= (2\pi) \int_{z=-\frac{\sqrt{3}}{2}a}^{z=+\frac{\sqrt{3}}{2}a} \mathrm{d}z \left[\frac{r^2}{2}\right]_{r=\frac{a}{2}}^{r=\sqrt{a^2-z^2}} \\
&= \pi \int_{z=-\frac{\sqrt{3}}{2}a}^{z=+\frac{\sqrt{3}}{2}a} \mathrm{d}z \left[a^2 - z^2 - \frac{a^2}{4}\right] \\
&= \pi \left[\frac{3}{4}a^2 z - \frac{z^3}{3}\right]_{z=-\frac{\sqrt{3}}{2}a}^{z=+\frac{\sqrt{3}}{2}a} \\
&= 2\pi a^3 \left[\frac{3}{4}\cdot\frac{\sqrt{3}}{2} - \frac{1}{3}\frac{\sqrt{3}^3}{8}\right] = \frac{2\sqrt{3}\pi a^3}{8}\left[3 - \frac{\sqrt{3}^2}{3}\right] = \frac{\sqrt{3}\pi}{2}a^3\,.
\end{aligned}$$

(3) Where is the center of mass of a right circular cone? Suppose the base has radius R and the cone has height H.

Solution: We set our coordinate system so that the z-axis is the axis of symmetry of the cone, and such that $z = 0$ at the *apex*. This way, if the point (r, θ, z) is *on the side of the cone* then the point (R, θ, H) is also on the side, and the right triangles with sides $z, r, \sqrt{z^2 + r^2}$ and $H, R, \sqrt{H^2 + R^2}$ are similar. The equation of the side of the cone is then $\frac{z}{H} = \frac{r}{R}$ (note that if we put $z = 0$ at the base, the equation would read $\frac{H-z}{H} = \frac{r}{R}$ which is fine, but complicates the algebra later). The equation of the base is simply $z = H$. We now set up integrals using either slicing method:

- The shadow on the xy plane is the disc $r \le R$ (coming from the base). A vertical "z-line" over a point (r, θ) on the xy plane will *begin* on the cone, and *end* on the base. And integral over the cone is then

$$\int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{r=0}^{r=R} r\,\mathrm{d}r \int_{z=\frac{H}{R}r}^{z=H} \mathrm{d}z f(r, \theta, z)\,.$$

In particular:

$$\begin{aligned}
V &= \int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{r=0}^{r=R} r\,\mathrm{d}r \int_{z=\frac{H}{R}r}^{z=H} \mathrm{d}z \cdot 1 \\
&= (2\pi) \int_{r=0}^{r=R} r\,\mathrm{d}r \left(H - \frac{H}{R}r\right) \\
&= 2\pi H \left[\frac{r^2}{2} - \frac{r^3}{3R}\right]_{r=0}^{r=R} = \frac{\pi R^2 H}{3}\,.
\end{aligned}$$

and

$$\begin{aligned}
\bar{z} &= \frac{1}{V} \int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{r=0}^{r=R} r\,\mathrm{d}r \int_{z=\frac{H}{R}r}^{z=H} \mathrm{d}z \cdot z \\
&= \frac{1}{V}(2\pi) \int_{r=0}^{r=R} r\,\mathrm{d}r \left[\frac{z^2}{2}\right]_{z=\frac{H}{R}r}^{z=H} \\
&= \frac{\pi}{V} \int_{r=0}^{r=R} r \left[H^2 - \frac{H^2}{R^2}r^2\right] \mathrm{d}r \\
&= \frac{\pi H^2}{V} \left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_{r=0}^{r=R} \\
&= \frac{\pi H^2}{\pi R^2 H/3} \left[\frac{R^2}{4}\right] = \frac{3}{4}H\,.
\end{aligned}$$

(amusingly this doesn't depend on R). The center-of-mass must also be on the axis of symmetry (on the z-axis), so we see that it is a point $\frac{3}{4}H$ away from the apex, so $\frac{1}{4}H$ above the base.

- Each slice of constant z is a disc of radius $\frac{R}{H}z$. The integral over the cone is then

$$\int_{z=0}^{z=H} \mathrm{d}z \int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{r=0}^{r=\frac{R}{H}z} r\,\mathrm{d}r f(r,\theta,z)\,.$$

In particular,

$$\begin{aligned} V &= \int_{z=0}^{z=H} \mathrm{d}z \int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{r=0}^{r=\frac{R}{H}z} r\,\mathrm{d}r \cdot 1 \\ &= \int_{z=0}^{z=H} \mathrm{d}z\,(2\pi)\left(\frac{R^2}{H^2}\frac{z^2}{2}\right) \\ &= \frac{\pi R^2}{H^2}\int_{z=0}^{z=H} z^2\,\mathrm{d}z \\ &= \frac{\pi R^2}{H^2}\cdot\frac{H^3}{3} = \frac{\pi R^2 H}{3}\,. \end{aligned}$$

and

$$\begin{aligned} \bar{z} &= \frac{1}{V}\int_{z=0}^{z=H} \mathrm{d}z \int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{r=0}^{r=\frac{R}{H}z} r\,\mathrm{d}r \cdot z \\ &= \frac{1}{V}\int_{z=0}^{z=H} \mathrm{d}z z\,(2\pi)\left(\frac{R^2}{H^2}\frac{z^2}{2}\right) \\ &= \frac{\pi R^2/H^2}{\pi R^2 H/3}\int_{z=0}^{z=H} z^3\,\mathrm{d}z \\ &= \frac{3}{H^3}\cdot\frac{H^4}{4} = \frac{3}{4}H\,. \end{aligned}$$