MATH 412: NOTE ON INFINITE-DIMENSIONAL VECTOR SPACES

Abstract. This is an explanatory note on what the basic definitions of linear algebra mean when the vector spaces are infinite-dimensional.

1. Three examples

1.1. **Finite dimensions: $\mathbb{R}^n$.** Consider the space $\mathbb{R}^n$ with its *standard basis* $\{\underline{e}_i\}_{i=1}^n$ ($\underline{e}_i$ is the vector with 1 at the ith position, and 0 elsewhere). Why is this set a basis? It is

(1) *Spanning*: Every vector in $\mathbb{R}^n$ can be written as a linear combination of basis elements, that is in the form$\sum_{i=1}^n a_i\underline{e}_i$ for some $a_i \in R$.
(2) *Independent*: If $\sum_{i=1}^n a_i\underline{e}_i = \underline{0}$ then all the a_i are zero; equivalently, the zero vector has a unique representation in the basis; equivalently, every vector can be uniquely represented.

1.2. **Infinite dimensions,explicit basis: $\mathbb{R}^{\oplus\mathbb{N}}$.** Let $\mathbb{R}^{\oplus\mathbb{N}}$denote the space of sequences $\underline{x} = (x_n)_{n=1}^{\infty}$ which have *finite support*, that is so that $x_n = 0$ from some point onward. This space is evidently a subset of $\mathbb{R}$, the space of all functions from $\mathbb{N}$ to $\mathbb{R}$ (recall that such functions are called "sequences"). So $\mathbb{R}^{\oplus\mathbb{N}}$ contains elements like $(1, 2, 3, 0, 0, 0, \cdots)$ and $(-1, 1, -1, 1, 0, 0, \cdots)$ but not the sequences $(1, 1, 1, 1, 1, \cdots)$ or $x_n = (-1)^n$.

Problem 1. Show that $\mathbb{R}^{\oplus\mathbb{N}}$ is a linear subspace of $\mathbb{R}$. In other words, show that it is non-empty and closed under addition and under scalar multiplication. (Hint: all you need to do is control the support of the sequence you get from the operation).

Let $\underline{e}_i$ now denote the sequence which is zero everywhere except for a 1 at the ith position. Once we understand what "linear combination" and "linear independence" mean we'll see that this is a basis for our space.

Claim 2 (Spanning). Every vector in $\mathbb{R}^{\oplus\mathbb{N}}$ can be expressed as a linear combination of the $\underline{e}_i$. Equivalently, the $\underline{e}_i$ span $\mathbb{R}^{\oplus}$.

Proof. Suppose that $\underline{x} = (x_n)_{n=1}^{\infty} \in \mathbb{R}^{\oplus}$, say that $x_n = 0$ if $n > N$. Then $\underline{x} = \sum_{i=1}^N x_i\underline{e}_i$. □

Note that the sum is *finite*. Every vector space comes with a binary operation $+$. With the use of induction (and since addition is associative) we can define finite sums – but infinite sums make no sense. Thus forming a linear combination using a vectors from a set S meant (1) choosing a finite subset of our set S (2) for each vector in the finite set, choosing a coefficient. This didn't appear in the first example because we could always enlarge the chosen finite subset to the whole set by extending the coefficients to be zero.

Claim 3 (Independence). The set $\{\underline{e}_i\}_{i=1}^{\infty}$ is linearly independent.

Proof. From the previous claim we learned what a *linear combination* is. So let $I \subset \mathbb{N}$ be a finite set (this means we will consider $\{\underline{e}_i\}_{i\in I}$, and suppose $a_i \in \mathbb{R}$are chosen, and suppose that

$$\sum_{i\in I} a_i \underline{e}_i = \underline{0}\,.$$

We need to show that all $a_i = 0$ – but the proof is now the same as in $\mathbb{R}^n$. We can do one of two things: either, directly take any $i \in I$ and consider the ith coordinate of the vector on the equality. From the RHS we see that it is zero. On the RHS this is a_i since $a_j\underline{e}_j$ has zero in the ith coordinate and addition in the space of sequences is done coordinate-by-coordinate. Alternatively, we could increase the set I to be the full interval $\{1, 2, \cdots, N\}$ (and setting $a_i = 0$ for $i \notin I$). This doesn't spoil the fact that now

$$\sum_{i=1}^{N} a_i \underline{e}_i = \underline{0}\,.$$

Finally, truncate the vectors to have length N. This shouldn't affect the equality, and that all $a_i = 0$ follows from the independence of the standard basis of $\mathbb{R}^N$. □

Claim 4. Every element of $\mathbb{R}^{\oplus\mathbb{N}}$has a *unique* representation as a combination of basis elements.

This is by far the most confusing point, and the main reason for this note, because with the naive notion of "unique representation" this seems false. Indeed, the zero vector has infinitely many representations, including $0\underline{e}_1$, $0\underline{e}_2$, $0\underline{e}_3$, etc.

Key: this objection is equally valid in finite dimensions. Why aren't $1\begin{pmatrix}1\\0\end{pmatrix}$ and $1\begin{pmatrix}1\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\end{pmatrix}$ considered distinct representations of $\begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{R}^2$ as a linear combination of elements of the standard basis?

- Answer: in fact, they are *the same representation!* – they only differ from each other in directions where the coefficient is zero. In finite dimensions some get around this by declaring every combination to include the whole basis, whether the basis elements were "redundant" in it or not, and this provided uniqueness. But that is not the only approach.

Exactly the same thing is true in $\mathbb{R}^{\oplus}$: we need to consider two representations to be the same if the only differ in terms which have a zero coefficients. There are two ways to do this:

(1) One approach is to "reduce" every combination by excising the vectors with zero coefficients. Such a representation is indeed unique: if you have $\sum_{i\in I} a_i\underline{e}_i = \sum_{j\in J} b_j\underline{e}_j$ and all a_i, b_j are non-zero, then subtract the two sides to get $\sum_{i\in I} a_i\underline{e}_i + \sum_{j\in J}(-b_j)\underline{e}_j = \underline{0}$. If some $i \in I \cap J$ then combine the two summands to get a combination where every basis vector appears once. Now if some i belongs to I but not j then in the resulting representation of zero there will be a term $a_i\underline{e}_i$, which is a contradiction to the proof of independence. If there is a j in J but not i, we'll get a term $(-b_j)\underline{e}_j$ and again it's a contradiction. Thus $I = J$, and our sum reads $\sum_{i\in I}(a_i - b_i)\underline{e}_i = \underline{0}$. Now independence forces $a_i = b_i$ and the combinations are the same.

(2) Another approach (mimicking the approach you are familiar with from finite dimensions) is to *formally* extend every combination to the whole basis, by assigning every coefficient a weight of zero. In other words, we define a

linear combination to mean a formal sum $\sum_{i=1}^{\infty} a_i \underline{e}_i$ where $a_i \in \mathbb{R}$ and all but finitely many are non-zero. The value of such a sum is defined to mean the value of any finite subsum containing all non-zero terms (note that this value doesn't depend on the choice of subsum, and that such choices exist). Now every vector has a representation using the whole basis, and uniqueness means the obvious thing (and has the same proof as in the finite-dimension case, coming from subtracting any two representations).

1.3. **Infinite dimensions, no explicit basis: $\mathbb{R}$.** Let's now look a little at the space $\mathbb{R}^{\mathbb{N}}$of all sequences. In this space consider the sequences $\underline{x}^{\alpha}$defined for each $\alpha \in \mathbb{R} \setminus \{0\}$ by $x_n^{\alpha} = \alpha^n$. They are linearly independent. Indeed, let $S \colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$be the *shift* operator $(S\underline{x})_n = x_{n+1}$, a linear map. Them $(S\underline{x}^{\alpha})_n = x_{n+1}^{\alpha} = \alpha^{(n+1)} = \alpha \cdot \alpha^n = \alpha\,(\underline{x}^{\alpha})_n$ so $\underline{x}^{\alpha}$ is an eigenvector of S with eigenvalue α. Since the eigenvalues are distinct, the eigenvectors are linearly independent, and we obtain a linearly independent set which is uncountably large. What does a linear combination of these sequences look like? We need to choose *finitely many* α, and some coefficients A_i, so a linear combination would look something like:

$$\underline{x} = \sum_{i=1}^{r} A_i \underline{x}^{\alpha_i} \,.$$

Example 5. Suppose a sequence $\underline{x}$ is suppose to solve a recursion relation: for each n it must hold that $a_k x_{n+k} + a_{k-1} x_{n+k-1} + \cdots + a_n x_n = 0$ (for example, the Fibonacci sequence satisfies $x_{n+2} - x_{n+1} - x_n = 0$). This is equivalent to saying that $\underline{x}$ is in the kernel of the linear map $\sum_{i=0}^{k} a_i S^i$. Now $\underline{x}^{\alpha}$ is in the kernel whenever α is a root of the polynomial equation $\sum_{i=0}^{k} a_i \alpha^i = 0$. In particular, we will show later in the course that if the polynomial has distinct roots than any solution to the recursion relation must be a linear combination of the $\underline{x}^{\alpha_i}$ where $\{\alpha_i\}$ are the roots of the polynomial $\sum_{i=0}^{k} a_i X^i$.

Problem 6. Does $\mathbb{R}^{\mathbb{N}}$have a basis?

This turns out to be a problem in the foundations of mathematics. In this course we choose the answer to be "yes".

2. General definitions

We now summarize what we saw in the examples above. For this let V be a vector space over a field F, and fix $S \subset V$. The proofs are all the same as in the finite-dimensional case and can also be found in my notes for Math 223 (see my website).

Definition 7. A *linear combination* using elements of S is a sum of the form $\sum_{i=1}^{r} a_i \underline{v}_i$ for some $r \geq 0$, where $a_i \in F$ and $\underline{v}_i \in S$. A vector $\underline{v} \in V$ is said to *depend* (linearly) on S if it is expressible as a linear combination of some vectors of S.

Remark: the empty sum is by definition $\underline{0}$. Note that different linear combinations have different lengths, and that some a_i may be zero.

Definition 8. The (linear) *span* of S is the set of all linear combinations of vectors from S (more formally, the set of all vectors expressible as linear combinations of vectors of S).

Lemma 9. $\mathrm{Span}_F(S) \subset V$ *is a subspace. In fact, it is the intersection of all subspaces containing* S*.*

Proof. Exercise. □

Definition 10. A (linear) *dependence* of vectors in S is a sum $\sum_{i=1}^{r} a_i \underline{v}_i = \underline{0}$ where $r \geq 1$, $a_i \in F$ are not all zero, and $\underline{v}_i \in S$ are *distinct.* The set S is said to be (linearly) *dependent* if it has a linear dependence, (linearly) *independent* if no such dependence exists.

Lemma 11 (Dependence and Independence)**.**
(1) *The set* S *is dependent if and only if there is some* $\underline{v} \in S$ *which depends linearly on* $S \setminus \{\underline{v}\}$*.*
(2) *The set* S *is independent if and only if, for every* $\underline{v} \in \mathrm{Span}_F(S)$*, there is a unique representation* $\underline{v} = \sum_{i=1}^{r} a_i \underline{v}_i$ *where* $\underline{v}_i \in S$ *are all distinct and the* $a_i \in F$ *are* all non-zero.

Proof. Exercise □

Remark 12. Note that the only representation of the zero vector is by the empty sum (the sum with $r = 0$).

Lemma 13 (Bases)**.**
(1) *(Maximal independent sets are spanning) Suppose* S *is linearly independent. Then* $\mathrm{Span}_F(S) = V$ *if and only* S *is a maximal independent set, in that for any* $\underline{v} \in V \setminus S$*,* $S \cup \{\underline{v}\}$ *is dependent.*
(2) *(Minimal spanning sets are independent) Suppose* S *spans* V*. Then* S *is independent if and only if for any* $\underline{v} \in S$*,* $S \setminus \{\underline{v}\}$ *does not span* V*.*

Proof. Exercise. □

Definition 14. If S is linearly independent and spans V is it called a *basis* of V.

Axiom 15 (Axiom of Choice)**.** *Every vector space has a basis.*

Proposition 16. *Every linearly independent subset of* V *is contained in a basis of* V*.*

Proof. See Problem Sets 2. □

Theorem 17. *Let* B_1, B_2 *be bases of* V*. Then there is a bijection* $f \colon B_1 \to B_2$*.*

Proof. Requires some cardinal arithmetic. □

Corollary 18. *The cardinality of a basis of* V *does not depend on the choice of basis.*

Definition 19. The dimension *of* V*, denoted* $\dim_F V$*, is the cardinality of any basis of* V*.*

Example 20. (You need to know something about cardinality for this example) The dimension of $\mathbb{R}^{\mathbb{N}}$is $\aleph = |\mathbb{R}|$.

Proof. In Section 1.3 we say that $\mathbb{R}^{\mathbb{N}}$ contains a linearly independent set enumerated by $\mathbb{R} \setminus \{0\}$. Since any independent set can be extended to a basis, $\dim_{\mathbb{R}} \mathbb{R}^{\mathbb{N}} \geq \aleph$. Conversely, $|\mathbb{R}| = \aleph = 2^{\aleph_0}$. Thus

$$\aleph = |\mathbb{R}| \leq \left|\mathbb{R}^{\mathbb{N}}\right| = \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = \aleph$$

so that $\left|\mathbb{R}^{\mathbb{N}}\right| = \aleph$. Since the cardinality of any basis is at most the cardinality of the space, we have $\dim_{\mathbb{R}} \left(\mathbb{R}^{\mathbb{N}}\right) \leq \left|\mathbb{R}^{\mathbb{N}}\right| = \aleph$ and we are done. □

Fact 21 (For amusement value only)**.** *It is consistent with the (ZF) axioms of set theory that either (1) every vector space has a basis; or (2) every subset of the real numbers is Lebesgue measurable. However, it is not possible that both (1) and (2) hold at the same time.*