

Math 412: Problem Set 3 (due 29/1/2014)

Practice

P1 Let $\underline{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\underline{u}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$, $\underline{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ as vectors in \mathbb{R}^3 .

- (a) Construct an explicit linear functional $\varphi \in (\mathbb{R}^3)'$ vanishing on $\underline{u}_1, \underline{u}_2$.
- (b) Show that $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ is a basis on \mathbb{R}^3 and find its dual basis.
- (c) Evaluate the dual basis at \underline{u} .

P2 Let V be n -dimensional and let $\{\varphi_i\}_{i=1}^m \in V'$.

- (a) Show that if $m < n$ there is a non-zero $\underline{v} \in V$ such that $\varphi_i(\underline{v}) = 0$ for all i . Interpret this as a statement about linear equations.
- (b) When is it true that for each $\underline{x} \in F^m$ there is $\underline{v} \in V$ such that for all i , $\varphi_i(\underline{v}) = x_i$?

Banach limits

Recall that $\ell^\infty \subset \mathbb{R}^\mathbb{N}$ denote the set of *bounded* sequences (the sequences \underline{a} such that for some M we have $|a_i| \leq M$ for all i). Let $S: \mathbb{R}^\mathbb{N} \rightarrow \mathbb{R}^\mathbb{N}$ be the *shift* map $(S\underline{a})_n = \underline{a}_{n+1}$. A subspace $U \subset \mathbb{R}^\mathbb{N}$ is *shift-invariant* if $S(U) \subset U$. If U is shift-invariant a function F with domain U is called *shift-invariant* if $F \circ S = F$ (example: the subset $c \subset \mathbb{R}^\mathbb{N}$ of convergent sequences is shift-invariant, as is the functional $\lim: c \rightarrow \mathbb{R}$ assigning to every sequence its limit).

P3 (Useful facts)

- (a) Show that ℓ^∞ is a subspace of $\mathbb{R}^\mathbb{N}$.
- (b) Show that $S: \mathbb{R}^\mathbb{N} \rightarrow \mathbb{R}^\mathbb{N}$ is linear and that $S(\ell^\infty) = \ell^\infty$.
- (c) Let $U \subset \mathbb{R}^\mathbb{N}$ be a shift-invariant subspace. Show that the set $U_0 = \{S\underline{a} - \underline{a} \mid \underline{a} \in U\}$ is a subspace of U .
- (d) In the case $U = \mathbb{R}^{\oplus \mathbb{N}}$ of sequences of finite support, show that $U_0 = U$.
- (e) Let Z be an auxiliary vector space. Show that $F \in \text{Hom}(U, Z)$ is shift-invariant iff F vanishes on U_0 .

1. Let $W = \{S\underline{a} - \underline{a} \mid \underline{a} \in \ell^\infty\} \subset \ell^\infty$. Let $\mathbb{1}$ be the sequences everywhere equal to 1.

- (a) Show that the sum $W + \mathbb{R}\mathbb{1} \subset \ell^\infty$ is direct and construct an S -invariant functional $\varphi: \ell^\infty \rightarrow \mathbb{R}$ such that $\varphi(\mathbb{1}) = 1$.
- (b) (Strengthening) For $\underline{a} \in \ell^\infty$ set $\|\underline{a}\|_\infty = \sup_n |a_n|$. Show that if $\underline{a} \in W$ and $x \in \mathbb{R}$ then $\|\underline{a} + x\mathbb{1}\|_\infty \geq |x|$. (Hint: consider the average of the first N entries of the vector $\underline{a} + x\mathbb{1}$).

SUPP Let $\varphi \in (\ell^\infty)'$ be shift-invariant, positive, and satisfy $\varphi(\mathbb{1}) = 1$. Show that $\liminf_{n \rightarrow \infty} a_n \leq \varphi(\underline{a}) \leq \limsup_{n \rightarrow \infty} a_n$ and conclude that the restriction of φ to c is the usual limit.

2. (“choose one”) Let $\varphi \in (\ell^\infty)'$ satisfy $\varphi(\mathbb{1}) = 1$. Let \underline{x} be the sequence $a_n = \frac{1+(-1)^n}{2}$.

- (a) Suppose that φ is shift-invariant. Show that $\varphi(\underline{x}) = \frac{1}{2}$.
- (b) Suppose that φ respects pointwise multiplication (if $z_n = x_n y_n$ then $\varphi(\underline{z}) = \varphi(\underline{x})\varphi(\underline{y})$). Show that $\varphi(\underline{a}) \in \{0, 1\}$.

Duality and bilinear forms

3. (The dual map) Let U, V, W be vector spaces, and let $T \in \text{Hom}(U, V)$, and let $S \in \text{Hom}(V, W)$.
- (a) (The abstract meaning of transpose) Suppose U, V be finite-dimensional with bases $\{\underline{u}_j\}_{j=1}^m \subset U$, $\{\underline{v}_i\}_{i=1}^n \subset V$, and let $A \in M_{n,m}(F)$ be the matrix of T in those bases. Show that the matrix of the dual map $T' \in \text{Hom}(V', U')$ with respect to the dual bases $\{\underline{u}'_j\}_{j=1}^m \subset U'$, $\{\underline{v}'_i\}_{i=1}^n \subset V'$ is the transpose tA .
- (b) Show that $(ST)' = T'S'$. It follows that ${}^t(AB) = {}^tB{}^tA$.
4. Let $F^{\oplus \mathbb{N}}$ denote the space of sequences of finite support. Construct a non-degenerate pairing $F^{\oplus \mathbb{N}} \times F^{\mathbb{N}} \rightarrow F$, giving a concrete realization of $(F^{\oplus \mathbb{N}})'$.
5. Let $C_c^\infty(\mathbb{R})$ be the space of smooth functions on \mathbb{R} with compact support, and let $D: C_c^\infty(\mathbb{R}) \rightarrow C_c^\infty(\mathbb{R})$ be the differentiation operator $\frac{d}{dx}$. For a reasonable function f on \mathbb{R} define a functional φ_f on $C_c^\infty(\mathbb{R})$ by $\varphi_f(g) = \int_{\mathbb{R}} fg \, dx$ (note that f need only be integrable, not continuous).
- (a) Show that if f is continuously differentiable then $D'\varphi_f = \varphi_{-Df}$.
- DEF For this reason one usually extends the operator D to the dual space by $D\varphi = -D'\varphi$, thus giving a notion of a “derivative” for non-differentiable and even discontinuous functions.
- (b) Let the “Dirac delta” $\delta \in C_c^\infty(\mathbb{R})'$ be the evaluation functional $\delta(f) = f(0)$. Express $(D\delta)(f)$ in terms of f .
- (c) Let φ be a linear functional such that $D'\varphi = 0$. Show that for some constant c , $\varphi = \varphi_{c1}$.

Supplement: The support of distributions

- A. (This is a problem in analysis unrelated to 412) Let $\varphi \in C_c^\infty(\mathbb{R})'$.
- DEF Let $U \subset \mathbb{R}$ be open. Say that φ is *supported away from* U if for any $f \in C_c^\infty(U)$, $\varphi(f) = 0$. The *support* $\text{supp}(\varphi)$ is the complement the union of all such U .
- (a) Show that $\text{supp}(\varphi)$ is closed, and that φ is supported away from $\mathbb{R} \setminus \text{supp}(\varphi)$.
- (b) Show that $\text{supp}(\delta) = \{0\}$ (see problem 5(b)).
- (c) Show that $\text{supp}(D\varphi) \subset \text{supp}(\varphi)$ (note that this is well-known for functions).
- (d) Show that $D\delta$ is not of the form φ_f for any function f .