Math 412: Problem Set 5 (due 14/2/2014)

Tensor products of maps

1. Let U,V be finite-dimensional spaces, and let $A \in \mathrm{End}(U)$, $B \in \mathrm{End}(V)$.
   (a) Show that $(\underline{u},\underline{v}) \mapsto (A\underline{u}) \otimes (B\underline{v})$ is bilinear, and obtain a linear map $A \otimes B \in \mathrm{End}(U \otimes V)$.
   (b) Suppose A,B are diagonable. Using an appropriate basis for $U \otimes V$, Obtain a formula for $\det(A \otimes B)$ in terms of $\det(A)$ and $\det(B)$.
   (c) Extending (a) by induction, show that $A^{\otimes k}$ induces maps $\mathrm{Sym}^k A \in \mathrm{End}(\mathrm{Sym}^k V)$ and $\bigwedge^k A \in \mathrm{End}(\bigwedge^k V)$.
   (**d) Show that the formula of (b) holds for all A,B.

2. Suppose $\frac{1}{2} \in F$, and let U be finite-dimensional. Construct isomorphisms
$$\{\text{ symmetric bilinear forms on } U\} \leftrightarrow \left(\mathrm{Sym}^2 U\right)' \leftrightarrow \mathrm{Sym}^2\left(U'\right).$$

Structure Theory

3. Let L be a lower-triangular square matrix with non-zero diagonal entries.
   (a) Give a "forward substitution" algorithm for solving $L\underline{x} = \underline{b}$ efficiently.
   (b) Give a formula for L^{-1}, proving in particular that L is invertible and that L^{-1} is again lower-triangular.
   RMK We'll see that if $\mathcal{A} \subset M_n(F)$ is a subspace containing the identity matrix and closed under matrix multiplication, then the inverse of any matrix in $\mathcal{A}$ belongs to cA, giving an abstract proof of the same result).

4. Let $U \in M_n(F)$ be *strictly upper-triangular*, that is upper triangular with zeroes along the diagonal. Show that $U^n = 0$ and construct such U with $U^{n-1} \neq 0$.

5. Let V be a finite-dimensional vector space, $T \in \mathrm{End}(V)$.
   (*a) Show that the following statements are equivalent:
   (1) $\forall \underline{v} \in V : \exists k \geq 0 : T^k\underline{v} = \underline{0}$; (2) $\exists k \geq 0 : \forall \underline{v} \in V : T^k\underline{v} = \underline{0}$.
   DEF A linear map satsfying (2) is called *nilpotent*. Example: see problem 4.
   (b) Find nilpotent $A,B \in M_2(F)$ such that $A+B$ isn't nilpotent.
   (c) Suppose that $A,B \in \mathrm{End}(V)$ are nilpotent and that A,B commute. Show that $A+B$ is nilpotent.

Supplementary problems

A. (The tensor algebra) Fix a vector space U.

(a) Extend the bilinear map $\otimes\colon U^{\otimes n}\times U^{\otimes m}\to U^{\otimes n}\otimes U^{\otimes m}\simeq U^{\otimes(n+m)}$ to a bilinear map $\otimes\colon \bigoplus_{n=0}^{\infty}U^{\otimes n}\times\bigoplus_{n=0}^{\infty}U^{\otimes n}\to\bigoplus_{n=0}^{\infty}U^{\otimes n}$.

(b) Show that this map $\otimes$ is associative and distributive over addition. Show that $1_F\in F\simeq U^{\otimes 0}$ is an identity for this multiplication.

DEF This algebra is called the *tensor algebra* $T(U)$.

(c) Show that the tensor algebra is *free:* for any F-algebra A and any F-linear map $f\colon U\to A$ there is a unique F-algebra homomorphism $\bar{f}\colon T(U)\to A$ whose restriction to $U^{\otimes 1}$ is f.

B. (The symmetric algebra). Fix a vector space U.

(a) Endow $\bigoplus_{n=0}^{\infty}\mathrm{Sym}^n U$ with a product structure as in 3(a).

(b) Show that this creates a commutative algebra $\mathrm{Sym}(U)$.

(c) Fixing a basis $\{\underline{u}_i\}_{i\in I}\subset U$, construct an isomorphism $F\left[\{x_i\}_{i\in I}\right]\to\mathrm{Sym}^* U$.

RMK In particular, $\mathrm{Sym}^*(U')$ gives a coordinate-free notion of "polynomial function on U".

(d) Let $I\lhd T(U)$ be the two-sided ideal generated by all elements of the form $\underline{u}\otimes\underline{v}-\underline{v}\otimes\underline{u}\in U^{\otimes 2}$. Show that the map $\mathrm{Sym}(U)\to T(U)/I$ is an isomorphism.

RMK When the field F has finite characteristic, the correct definition of the symmetric algebra (the definition which gives the universal property) is $\mathrm{Sym}(U)\stackrel{\text{def}}{=}T(U)/I$, not the space of symetric tensors.

C. Let V be a (possibly infinite-dimensional) vector space, $A\in\mathrm{End}(V)$.

(a) Show that the following are equivalent for $\underline{v}\in V$: (1) $\dim_F\mathrm{Span}_F\{A^n\underline{v}\}_{n=0}^{\infty}<\infty$;
(2) there is a finite-dimensional subspace $\underline{v}\in W\subset V$ such that $AW\subset W$.

DEF Call such $\underline{v}$ *locally finite*, and let V_{fin} be the set of locally finite vectors.

(b) Show that V_{fin} is a subspace of V.

(c) A A is called *locally nilpotent* for every $\underline{v}\in V$ there is $n\geq 0$ such that $A^n\underline{v}=\underline{0}$ (condition (1) of 5(a)). Find a vector space V and a locally nilpotent map $A\in\mathrm{End}(V)$ which is not nilpotent.

(*d) A is called *locally finite* if $V_{\text{fin}}=V$, that is if every vector is contained in a finite-dimensional A-invariant subspace. Find a space V and locally finite linear maps $A,B\in\mathrm{End}(V)$ such that $A+B$ is not locally finite.