Math 412: Problem set 7 (due 10/3/2014)

Practice

P1. Find the characteristic and minimal polynomial of each matrix:

$$\begin{pmatrix} 1 & 1 & & 0 \\ & 1 & 0 & \\ 0 & 0 & 1 & \\ 0 & 0 & & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

P2. Show that $\begin{pmatrix} 0 & 1 & \alpha \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ are similar. Generalize to higher dimensions.

P3. Let V be finite-dimensional and let $T \in \mathrm{End}_F(V)$ satisfy $T^k = \mathrm{Id}$. Then T is diagonable.

The Jordan Canonical Form

1. For each of the following matrices, (i) find the characteristic polynomial and eigenvalues (over the complex numbers), (ii) find the eigenspaces and generalized eigenspaces, (iii) find a Jordan basis and the Jordan form.

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   RMK I suggest computing by hand first even if you later check your answers with a CAS.

2. Suppose the characteristic polynomial of T is $x(x-1)^3(x-3)^4$.
   (a) What are the possible minimal polynomials?
   (b) What are the possible Jordan forms?

3. Let $T, S \in \mathrm{End}_F(V)$.
   (a) Suppose that T, S are similar. Show that $m_T(x) = m_S(x)$.
   (b) Prove or disprove: if $m_T(x) = m_S(x)$ and $p_T(x) = p_S(x)$ then T, S are similar.

4. Let F be algebraically closed of characteristic zero. Show that every $g \in \mathrm{GL}_n(F)$ has a square root, that is $g = h^2$ for some $h \in \mathrm{GL}_n(F)$.

5. Let V be finite-dimensional, and let $\mathcal{A} \subset \mathrm{End}_F(V)$ be an F-subalgebra, that is a subspace containing the identity map and closed under multiplication (composition). Suppose that $T \in \mathcal{A}$ is invertible in $\mathrm{End}_F(V)$. Show that $T^{-1} \in \mathcal{A}$.

(extra credit problem on reverse)

Extra credit

6. (The additive Jordan decomposition) Let V be a finite-dimensional vector space, and let $T \in \mathrm{End}_F(V)$.

   DEF An *additive Jordan decomposition* of T is an expression $T = S + N$ where $S \in \mathrm{End}_F(V)$ is diagonable, $N \in \mathrm{End}_F(V)$ is nilpotent, and S, N commute.

   (a) Suppose that F is algebraically closed. Separating the Jordan form into its diagonal and off-diagonal parts, show that T has an additive Jordan decomposition.

   (b) Let $S, S' \in \mathrm{End}_F(V)$ be diagonable and suppose that S, S' commute. Show that $S + S'$ is diagonable.

   (c) Show that a nilpotent diagonable linear transformation vanishes.

   (d) Suppose that T has two decompositions as in (a) (into commuting diagonable and nilpotent parts) $T = S + N = S' + N'$. Show that $S = S'$ and $N = N'$.

Supplementary problems

A. (extension of scalars for linear algebra) Let $F \subset K$ be fields and let V be an F-vectorspace. Let $V_K = K \otimes_F V$, where we consider K as an F-vectorspace in the natural way.

   (a) Show that setting $\alpha(u \otimes \underline{v}) = (\alpha u) \otimes \underline{v}$ extends to a map $K \times V_K \to V_K$ satisfying the axioms of scalar multiplication for a K-vectorspace and compatible with the structure of $V_K$as an F-vectorspace coming from the tensor product.

   (b) Let $\{\underline{v}_i\}_{i \in I} \subset V$ be a set of vectors. Show that it is linearly independent (resp. spanning) iff $\{1_K \otimes \underline{v}_i\}_{i \in I} \subset V_K$ is linearly independent (resp. spanning).

   RMK This is how we show that the minimal polynomial does not depend on the field.

   (c) For $T \in \mathrm{End}_F(V)$ let $T_K = \mathrm{Id}_K \otimes_F T \in \mathrm{End}_F(V_K)$ be the tensor product map. Show that T_K is in fact K-linear.

   (d) Show that $T_K \in \mathrm{End}_K(V_K)$ is the unique K-linear map such that for any basis $\{\underline{v}_i\}_{i \in I} \subset V$, the matrix of T_K in the basis $\{1_K \otimes_F \underline{v}_i\}_{i \in I}$ is the matrix of T in the basis $\{\underline{v}_i\}$ (identification of the matrices under the inclusion $F \subset K$).

B. (conjugacy classes in $\mathrm{GL}_n(F)$) Let F be a field, and let $G = \mathrm{GL}_n(F)$.

   (a) Construct a bijection between conjugacy classes in G and certain Jordan forms. Note that the spectrum can lie in an extension field.

   (b) Enumerate the conjugacy classes in $\mathrm{GL}_2(\mathbb{F}_p)$.

   (c) Enumerate the conjugacy classes of $\mathrm{GL}_3(\mathbb{F}_p)$.