Math 412: Problem set 9, due 26/3/2014

1. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Let $\underline{v}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
   (a) Find S invertible and D diagonal such that $A = S^{-1}DS$.
   – Prove for yourself the formula $A^k = S^{-1}D^kS$.
   (b) Find a formula for $\underline{v}_k = A^k\underline{v}_0$, and show that $\frac{\underline{v}_k}{\|\underline{v}_k\|}$ converges for any norm on $\mathbb{R}^2$.
   RMK You have found a formula for Fibbonacci numbers (why?), and have shown that the real number $\frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^n$ is exponentially close to being an integer.
   RMK This idea can solve any *difference equation*. We will also apply this to solving *differential* equations.

2. Let $A = \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}$ with $z \in \mathbb{C}$.
   (a) Find (and prove) a simple formula for the entries of A^n.
   (b) Use your formula to decide the set of z for which $\sum_{n=0}^{\infty} A^n$ converge, and give a formula for the sum.
   (c) Show that the sum is $(\mathrm{Id} - A)^{-1}$ when the series converges.

3. For each n construct a projection $E_n\colon \mathbb{R}^2 \to \mathbb{R}^2$ of norm at least n ($\mathbb{R}^n$ is equipped with the Euclidean norm unless specified otherwise).
   RMK Prove for yourself that the norm of an *orthogonal* projection is 1.

Supplementary problems

A. Consider the map $\mathrm{Tr}\colon M_n(F) \to F$.
   (a) Show that this is a continuous map.
   (b) Find the norm of this map when $M_n(F)$ is equipped with the $L^1 \to L^1$ operator norm (see PS8 Problem 2(a)).
   (c) Find the norm of this map when $M_n(F)$ is equipped with the Hilbert–Schmidt norm (see PS8 Problem 4).
   (*d) Find the norm of this map when $M_n(F)$ is equipped with the $L^p \to L^p$ operator norm. Find the matrices A with operator norm 1 and trace maximal in absolute value.

B. Call $T \in \mathrm{End}_F(V)$ *bounded below* if there is $K > 0$ such that $\|T\underline{v}\| \geq K\|\underline{v}\|$ for all $\underline{v} \in V$.
   (a) Let T be boudned below. Show that T is invertible, and that T^{-1} is a bounded operator.
   (*b) Suppose that V is finite-dimensional. Show that every invertible map is bounded below.