Math 539: Problem Set 0 (due 15/1/2013)

Real analysis

1. Some asymptotics
   (a) Let f,g be functions such that $f(x),g(x) > 2$ for x large enough. Show that $f \ll g$ implies $\log f \ll \log g$. Give a counterexample under the weaker hypothesis $f(x),g(x) > 1$.
   (b) For all $A > 0$, $0 < b < 1$ and $\varepsilon > 0$ show that for $x \geq 1$,

$$\log^A x \ll \exp\left(\log^b x\right) \ll x^{\varepsilon}.$$

2. Set $\log_1 x = \log x$ and for x large enough, $\log_{k+1} x = \log(\log_k x)$. Fix $\varepsilon > 0$.
   (PRAC) Find the interval of definition of $\log_k x$. For the rest of the problem we suppose that $\log_k x$ is defined at N.
   (a) Show that $\sum_{n=N}^{\infty} \frac{1}{n \log n \log_2 n \cdots \log_{k-1} n (\log_k n)^{1+\varepsilon}}$ converges.
   (b) Show that $\sum_{n=N}^{\infty} \frac{1}{n \log n \log_2 n \cdots \log_{k-1} n (\log_k n)^{1-\varepsilon}}$ diverges.

3. (Stirling's formula)
   (a) Show that $\int_{k-1/2}^{k+1/2} \log t \, \mathrm{d}t - \log k = O(\frac{1}{k^2})$.
   (b) Show that there is a constant C such that

$$\log(n!) = \sum_{k=1}^{n} \log k = \left(n + \frac{1}{2}\right) \log n - n + C + O\left(\frac{1}{n}\right).$$

   RMK $C = \frac{1}{2}\log(2\pi)$, but this is largely irrelevant.

4. Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be sequences with partial sums $A_n = \sum_{k=1}^{n} a_k$, $B_n = \sum_{k=1}^{n} b_k$.
   (a) (Abel summation formula) $\sum_{n=1}^{N} a_n b_n = A_N b_N - \sum_{n=1}^{N-1} A_n (b_{n+1} - b_n)$
   – (Summation by parts formula) Show that $\sum_{n=1}^{N} a_n B_n = A_N B_N - \sum_{n=1}^{N-1} A_n b_{n+1}$.
   (b) (Dirichlet's criterion) Suppose that $\{A_n\}_{n=1}^{\infty}$ are uniformly bounded and that $b_n \in \mathbb{R}_{>0}$ decrease monotonically to zero. Show that $\sum_{n=1}^{\infty} a_n b_n$ converges.

Supplementary problem: Review of Arithmetic functions

A.
   (a) The set of arithmetic functions with pointwise addition and Dirichlet convolution forms a commutative ring. The identity element is the function $\delta(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$.
   (b) f is invertible in this ring iff $f(1)$ is invertible in $\mathbb{C}$.
   (c) If f,g are multiplicative so is $f * g$.
   DEF $I(n) = 1$, $N(n) = n$, $\varphi(n) = \left|(\mathbb{Z}/n\mathbb{Z})^{\times}\right|$, $\mu(n) = (-1)^r$ if n is a product of $r \geq 0$ distinct primes, $\mu(n) = 0$ otherwise (i.e. if n is divisible by some p^2).
   (d) Show that $I * \mu = \delta$ by explicitly evaluating the convolution at $n = p^m$ and using (c).
   (e) Show that $\varphi * I = N$: (i) by explcitly evaluating the convolution at $n = p^m$ and using (c); (ii) by a combinatorial argument.

Supplementary problems: the Mellin transform and the Gamma function

For a function ϕ on $(0,\infty)$ its *Mellin transform* is given by $\mathcal{M}\phi(s) = \int_0^\infty \phi(x)x^s \frac{dx}{x}$ whenver the integral converges absolutely.

B. Let ϕ be a bounded measurable function on $(0,\infty)$.

(a) Suppose that for some $\alpha > 0$ we have $\phi(x) = O(x^{-\alpha})$ as $x \to \infty$. Show that the $\mathcal{M}\phi$ defines a holomorphic function in the strip $0 < \Re(s) < \alpha$.

For the rest of the problem assume that $\phi(x) = O(x^{-\alpha})$ holds for all $\alpha > 0$.

(b) Suppose that ϕ is smooth in some interval $[0,b]$ (that is, there $b > 0$ and is a function $\psi \in C^\infty([0,b])$ such that $\psi(x) = \phi(x)$ with $0 < x \le b$). Show that $\tilde{\phi}(s)$ extends to a meromorphic function in $\mathbb{C}$, with at most simple poles at $-m$, $m \in \mathbb{Z}_{\ge 0}$ where the residues are $\frac{\phi^{(m)}(0)}{m!}$ (in particular, if this derivative vanishes there is no pole).

(c) Extend the result of (b) to ϕ such that $\phi(x) - \sum_{i=1}^{r} \frac{a_i}{x^i}$ is smooth in an interval $[0,b]$.

(d) Let $\Gamma(s) = \int_0^\infty e^{-t}t^s \frac{dt}{t}$. Show that $\Gamma(s)$ extends to a meromorphic function in C with simple poles at $\mathbb{Z}_{\le 0}$ where the residues are 1.

C. (The Gamma function) Let $\Gamma(s) = \int_0^\infty e^{-t}t^s \frac{dt}{t}$, defined initially for $\Re(s) > 0$.

FACT A standard integration by parts shows that $s\Gamma(s) = \Gamma(s+1)$ and hence $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}_{\ge 1}$.

(a) Let $Q_N(s) = \int_0^N \left(1 - \frac{x}{N}\right)^N x^s \frac{dx}{x}$. Show that $Q_N(s) = \frac{N!}{s(s+1)\cdots(s+N)}N^s$. Show that $0 \le \left(1 - \frac{x}{N}\right)^N \le e^{-x}$ holds for $0 \le x \le N$, and conclude that $\lim_{N\to\infty} \frac{N!}{s(s+1)\cdots(s+N)}N^s = \Gamma(s)$ for on $\Re s > 0$ (for a quantitative argument show instead $0 \le e^{-x} - \left(1 - \frac{x}{N}\right)^N \le \frac{x^2}{N}e^{-x}$)

(b) Define $f(s) = se^{\gamma s}\prod_{n=1}^\infty \left(1 + \frac{s}{n}\right)e^{-s/n}$ where $\gamma = \lim_{n\to\infty}\left(\sum_{i=1}^n \frac{1}{i} - \log n\right)$ is Euler's constant. Show that the product converges locally uniformly absolutely and hence defines an entire function in the complex plane, with zeros at $\mathbb{Z}_{\le 0}$. Show that $f(s+1) = \frac{1}{s}f(s)$.

(c) Let $P_N(s) = se^{\gamma s}\prod_{n=1}^N \left(1 + \frac{s}{n}\right)e^{-s/n}$. Show that for $\alpha \in (0,\infty)$,$\lim_{N\to\infty} Q_N(\alpha)P_N(\alpha) = 1$ and conclude (without using problem B) that $\Gamma(s)$ extends to a meromorphic function in $\mathbb{C}$ with simple poles at $\mathbb{Z}_{\le 0}$, that $\Gamma(s) \ne 0$ for all $s \in \mathbb{C} \setminus \mathbb{Z}_{\le 0}$ and that the Weierstraß product representation

$$\Gamma(s) = \frac{e^{-\gamma s}}{s}\prod_{n=1}^\infty \left(1 + \frac{s}{n}\right)^{-1} e^{s/n}$$

holds.

(d) Let $\digamma(s) = \frac{\Gamma'(s)}{\Gamma(s)}$ be the Digamma function. Using the Euler–Maclaurin summation formula $\sum_{n=0}^{n=N} f(n) = \int_0^N f(x)\,dx + \frac{1}{2}(f(0) + f(N)) + \frac{1}{12}(f'(0) - f'(N)) + R$, with $|R| \le \frac{1}{12}\int_0^N |f''(x)|\,dx$, show that if $|s| > \delta$ and $-\pi + \delta \le \arg(s) \le \pi + \delta$ then

$$\digamma(s) = \log s - \frac{1}{2s} + O_\delta\left(|s|^{-2}\right).$$

Integrating on an appropriate contour, obtain *Stirling's Approximation*: there is a constant c such that for $\mathfrak{a}_{\mathbb{R}}^*(s)$ in the given range,

$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + c + O_\delta\left(\frac{1}{|s|}\right).$$

(e) Show *Euler's reflection formula*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}\,.$$

Conclude that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and hence that $\int_{-\infty}^{+\infty} e^{-\alpha x^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}$.

(f) Setting $s = \frac{1}{2} + it$ in the reflection formula and letting $t \to \infty$, show that $c = \frac{1}{2}\log(2\pi)$ in Stirling's Approximation.

(g) Show *Legendre's duplication formula*

$$\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}) = \sqrt{\pi}2^{1-s}\Gamma(s)\,.$$