# Math 100: Snell's Law

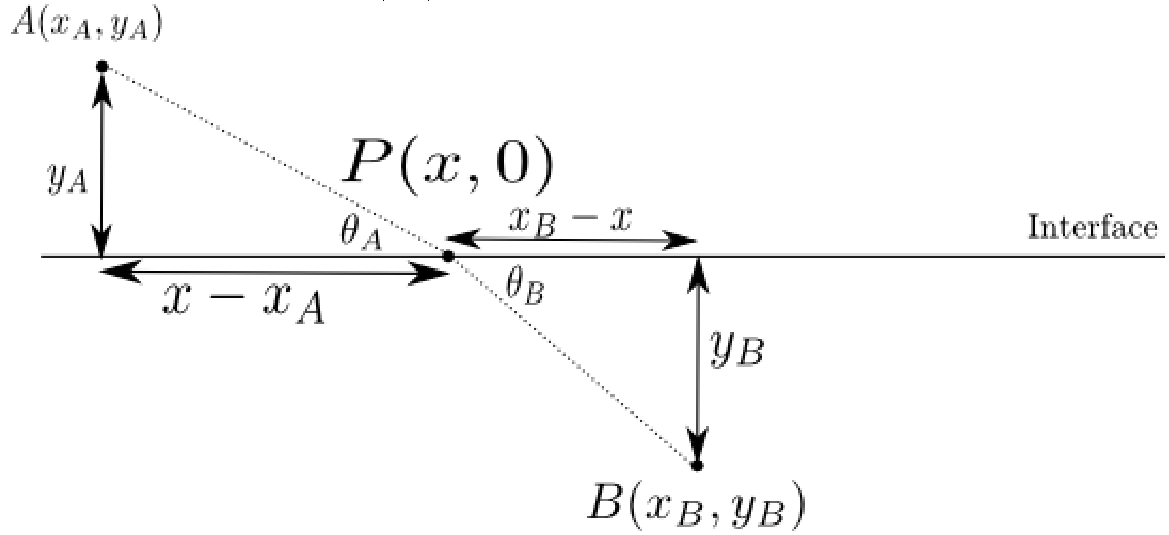
November 19, 2014

**Problem:**

It is known (*Fermat's principle*) that rays of light travel on paths of *minimum time*. When the speed of light is constant, this is the same as shortest paths or straight lines. But when the speed of line changes, this is more complicated. Suppose light travels from a point $A$ in a medium where the speed of light is $v_A$ to a point $B$ in another medium where the speed of light is $v_B$. Where does it cross?

**Solution:**

Put the $x$-axis along the interface, and suppose the point $A$ is at $(x_A, y_A)$ and the point $B$ is at $(x_B, -y_B)$. Suppose the crossing point is at $P\,(x, 0)$. We obtain the following setup:




Then the travel distances are $|AP| = \sqrt{(x - x_A)^2 + y_A^2}$, $|BP| = \sqrt{(x - x_B)^2 + y_B^2}$. The total travel time is therefore

$$f(x) = \frac{\sqrt{(x - x_A)^2 + y_A^2}}{v_A} + \frac{\sqrt{(x - x_B)^2 + y_B^2}}{v_B}.$$

Since $y_A^2, y_B^2 > 0$ this function is defined and differentiable for all $x$. To analyze it set $f_A(x) = \sqrt{(x - x_A)^2 + y_A^2}$. Then

$$f_A(x) = \frac{|x|}{\sqrt{x^2}}\sqrt{(x - x_A)^2 + y_A^2} = |x|\sqrt{(1 - \frac{x_A}{x})^2 + \frac{y_A^2}{x^2}}.$$

Now let $x \to +\infty$ or $x \to -\infty$. In either case, $\frac{1}{x} \to 0$ so the term under the root converges to $\sqrt{(1-0)^2 + 0} = 1$. It follows that $\lim_{x\to\pm\infty} f_A(x) = \infty$. Of course, the same holds for $f_B(x) = \sqrt{(x - x_B)^2 + y_B^2}$ and we get that

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \left( \frac{f_A(x)}{v_A} + \frac{f_B(x)}{v_B} \right) = \infty\,.$$

It follows that $f(x)$ achieves its minimum at some finite $x$, which is therefore a zero of the derivative. We now calculate:

$$\begin{aligned} f_A'(x) &= \frac{2(x-x_A)}{2\sqrt{(x-x_A)^2+y_A^2}} & &= \frac{x-x_A}{\sqrt{(x-x_A)^2+y_A^2}} = \cos\theta_A \\ f_B'(x) &= \frac{x-x_B}{\sqrt{(x-x_B)^2+y_B^2}} & &= \frac{x-x_B}{\sqrt{(x-x_B)^2+y_B^2}} = -\cos\theta_B \end{aligned}$$

where $\theta_A, \theta_B$ are the angles of refraction in the following diagram:

We conclude that

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left( \frac{f_A(x)}{v_A} + \frac{f_B(x)}{v_B} \right) = \frac{\cos\theta_A}{v_A} - \frac{\cos\theta_B}{v_B}\,.$$

In particular, $f'(x) = 0$ occurs if and only if $\frac{\cos\theta_A}{v_A} - \frac{\cos\theta_B}{v_B} = 0$ that is if

$$\frac{\cos\theta_A}{v_A} = \frac{\cos\theta_B}{v_B}$$

which we rearrange as

$$\frac{\cos\theta_A}{\cos\theta_B} = \frac{v_A}{v_B}\,.$$