MATH 100 – SOLUTIONS TO WORKSHEET 11
LOGARITHMIC DIFFERENTIATION, APPLICATIONS

1. Logarithmic Differentiation

(1) Differentiate.

(a) x^x

Solution: We use logarithmic differentiation: $f' = f \cdot (\log f)'$. Here $\log(x^x) = x \log x$ so by the product rule $(\log(x^x))' = \log x + \frac{x}{x} = 1 + \log x$ and

$$\boxed{(x^x)' = x^x (1 + \log x)\ .}$$

(b) $(\log x)^{\cos x}$

Solution: We use logarithmic differentiation: $f' = f \cdot (\log f)'$. Here $\log\left((\log x)^{\cos x}\right) = \cos x \cdot \log\log x$ so by the product rule and the chain rule, $\left(\log\left((\log x)^{\cos x}\right)\right)' = -\sin x \log\log x + \cos x (\log\log x)' = -\sin x \log\log x + \cos x \frac{1}{\log x}\frac{1}{x}$ and

$$\boxed{\left((\log x)^{\cos x}\right)' = (\log x)^{\cos x}\left(\frac{\cos x}{x \log x} - \sin x \log\log x\right)\ .}$$

(c) (Final 2014) Let $y = x^{\log x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x only.

Solution: We use logarithmic differentiation: $\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \frac{\mathrm{d}}{\mathrm{d}x}(\log y)$. Here $\log y = \log(x^{\log x}) = \log x \cdot \log x = \log^2 x$ so by the chain rule $\frac{\mathrm{d}\log y}{\mathrm{d}x} = 2\log x \frac{1}{x}$ and

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \frac{2\log x}{x} = x^{\log x} \cdot \frac{2\log x}{x} = 2x^{\log x - 1}\log x\,.}$$

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2. Applications

(1) The position of a particle at time t is given by $f(t) = \frac{1}{\pi}\sin(\pi t)$.
 (a) Find the velocity at time t, and specifically at $t = 3$.
 Solution: $v(t) = \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{1}{\pi}(\pi\cos(\pi t)) = \cos(\pi t)$ so that $v(3) = \cos(3\pi) = \cos(\pi) = -1$.
 (b) When is the particle accelerating? Decelerating?
 Solution: $a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = -\pi\sin(\pi t)$ so the acceleration is positive when $\sin(\pi t) < 0$ (when $t \in (2k-1, 2k)$ for some $k \in \mathbb{Z}$) and positive when $\sin(\pi t) > 0$ (when $t \in (2k, 2k+1)$ for some $k \in \mathbb{Z}$). However, in everyday language we usually say "accelerate" when the *speed* increases, not when the velocity increases (these are different when the velocity is negative!); you may want to work out the times when $a(t), v(t)$ have the same sign ("acceleration") and $a(t), v(t)$ have opposite signs ("decceleration").

(2)
 (a) Water is filling a cylindrical container of radius $r = 10$cm. Suppose that at time t the height of the water is $(t + t^2)$ cm. How fast is the volume growing?
 Solution: The volume of a cylinder of height h is $V = \pi r^2 h$ so we have $V(t) = \pi r^2(t + t^2)$ and (since $r = 10$ for us)

$$\boxed{\frac{\mathrm{d}V}{\mathrm{d}t} = 100\pi(1 + 2t)\,.}$$

 (b) A rocket is flying in space. The momentum of the rocket is given by the formula $p = mv$, where m is the mass and v is the velocity. At a time where the mass of the rocket is $m = 1000$kg and its velocity is $v = 500\frac{\mathrm{m}}{\mathrm{sec}}$ the rocket is accelerating at the rate $a = 20\frac{\mathrm{m}}{\mathrm{sec}^2}$ and losing mass at the rate $10\frac{\mathrm{kg}}{\mathrm{sec}}$. Find the rate of change of the momentum with time.
 Solution: Differentiating $p = mv$ using the product rule we find:

$$\boxed{F = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}t}v + m\frac{\mathrm{d}v}{\mathrm{d}t} = -10\cdot 500 + 1000\cdot 20 = 15,000\frac{\mathrm{kgm}}{\mathrm{s}^2}}$$

(3) A ball is falling from rest in air. Its height at time t is given by

$$h(t) = H_0 - gt_0\left(t + t_0 e^{-t/t_0} - t_0\right)$$

where H_0 is the initial height and t_0 is a constant.
 (a) Find the velocity of the ball.

$$\begin{aligned} v(t) &= \frac{\mathrm{d}h}{\mathrm{d}t} \\ &= 0 - gt_0\left(1 + t_0 e^{-t/t_0}\left(-\frac{1}{t_0}\right) - 0\right) \\ &= \boxed{-gt_0\left(1 - e^{-t/t_0}\right)\,.} \end{aligned}$$

 (b) Find the acceleration.

$$\begin{aligned} a(t) &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= -gt_0\left(0 - e^{-t/t_0}\left(-\frac{1}{t_0}\right)\right) \\ &= \boxed{-ge^{-t/t_0}\,.} \end{aligned}$$

 (c) Find $\lim_{t\to\infty} v(t)$.

$$\begin{aligned} \lim_{t\to\infty} v(t) &= \lim_{t\to\infty}\left(-gt_0\left(1 - e^{-t/t_0}\right)\right) \\ &= -gt_0\left(1 - \lim_{t\to\infty} e^{-t/t_0}\right) = -gt_0(1 - 0) \\ &= \boxed{-gt_0\,.} \end{aligned}$$