# MATH 100 – SOLUTIONS TO WORKSHEET 13
# RELATED RATES AND THE LINEAR APPROXIMATION

## 1. Related rates

(1) A particle is moving along the curve $y^2 = x^3 + 2x$. When it passes the point $\left(1, \sqrt{3}\right)$ we have $\frac{\mathrm{d}y}{\mathrm{d}t} = 1$. Find $\frac{\mathrm{d}x}{\mathrm{d}t}$.

**Solution 1** Differentiate using the chain rule to find $2y\frac{\mathrm{d}y}{\mathrm{d}t} = \left(3x^2+2\right)\frac{\mathrm{d}x}{\mathrm{d}t}$, so

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2y}{3x^2+2}\frac{\mathrm{d}y}{\mathrm{d}t}\,,$$

and at the given instand we get

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2\cdot\sqrt{3}}{2\cdot 1^2+2}\cdot 1 = \frac{2\sqrt{3}}{5}\,.$$

**Solution 2** Implicit differentiation gives $2y\frac{\mathrm{d}y}{\mathrm{d}x} = \left(3x^2+2\right)$, so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2+2}{2y}\,,$$

and at the given point $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{2\sqrt{3}}$.

Finally, by the chain rule $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}$ so

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}y/\,\mathrm{d}x} = \frac{1}{5/(2\sqrt{3})} = \frac{2\sqrt{3}}{5}\,.$$

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(2) Two ships are travelling near an island. The first is located 20km due west of it and is moving due north at 5km/h. The second is located 15km due south of it and is moving due south at 7km/h. How fast is the distance between the ships changing?

**Solution**

(a) Draw picture (skipped); put co-ordinate system centered at the island with $y$-axis going north-south and $x$-axis going east-west.

(b) Parametrize: the first ship is moving north-south, so write its location as $(-20, y_1(t))$. The second ship is moving north-south too, so parametrize its location is $(0, y_2(t))$. Write $D$ for the distance between the ships.

(c) Find relations: By Pythagoras, we have:

$$D^2 = (-20-0)^2 + (y_1-y_2)^2\,.$$

(d) Calculus: Differentiating with respect to time and using the chain rule, we find

$$\begin{aligned} 2D\cdot\frac{\mathrm{d}D}{\mathrm{d}t} &= 0 + \frac{\mathrm{d}}{\mathrm{d}t}\left[(y_1-y_2)^2\right] = 2\,(y_1-y_2)\,\frac{\mathrm{d}}{\mathrm{d}t}\,[y_1-y_2] \\ &= 2\,(y_1-y_2)\left(\frac{\mathrm{d}y_1}{\mathrm{d}t} - \frac{\mathrm{d}y_2}{\mathrm{d}t}\right) \end{aligned}$$

(e) Solve: We conclude that

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{y_1-y_2}{D}\left(\frac{\mathrm{d}y_1}{\mathrm{d}t} - \frac{\mathrm{d}y_2}{\mathrm{d}t}\right)\,.$$

*Date*: 22/10/2015, Worksheet by Lior Silberman.

At the given time we have $y_1 = 0$, $y_2 = -15$, $D = \sqrt{20^2 + 15^2} = \sqrt{5^2(4^2 + 3^2)} = 5\sqrt{25} = 25$, $\frac{\mathrm{d}y_1}{\mathrm{d}t} = 5$ (moving north!), $\frac{\mathrm{d}y_2}{\mathrm{d}t} = -7$ (moving south!) so

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{0-(-15)}{25}(5-(-7)) = \frac{15 \cdot 12}{25} = \frac{36}{5}\mathrm{km/h}$$

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(3) The same setting, but now the first ship is moving toward the island.

**Solution**

(a) Draw picture: same as before

(b) Parametrize: the first ship is now moving east-west, so write its location as $(x(t), 0)$. The second ship is still moving north-south, so parametrize its location is $(0, y(t))$. Write $D$ for the distance between the ships.

(c) Find relations: By Pythagoras, we have:

$$D^2 = x^2 + y^2 \,.$$

(d) Calculus: Differentiating with respect to time and using the chain rule, we find

$$2D\frac{\mathrm{d}D}{\mathrm{d}t} = 2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t}$$

(e) Solve: We conclude that

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{1}{D}\left(x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t}\right) .$$

At the given time we have $x = -20$, $y = -15$, $D = \sqrt{20^2 + 15^2} = 25$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 5$ (moving east!), $\frac{\mathrm{d}y}{\mathrm{d}t} = -7$ (moving south!) so

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{1}{25}(-20 \cdot 5 + (-15)(-7)) = \frac{105-100}{25} = \frac{1}{5}\mathrm{km/h} \,.$$

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(4) A conical drain is 6m tall and has radius 1m at the top.

(a) The drain is clogged, and is filling up with rain water at the rate of $5\mathrm{m}^3/\mathrm{min}$. How fast is the water rising when its height is 5m?

**Solution**

(i) Draw picture: (skipped) the water inside the cone fills a conical shape.

(ii) Parametrize: Say the drain has height $H$ and radius $R$ at the top. The water has height $h$ and radius $r$ at the top. Say the water volume is $V$.

(iii) Find relations: We know that $V = \frac{1}{3}\pi r^2 h$. Taking a vertical cross-section of the cone (draw picture!) we get from similar triangles that

$$\frac{r}{h} = \frac{R}{H} \,.$$

The problem involves heights so we'd like to eliminate $r$. Setting $r = \frac{R}{H}h$ we see that

$$\boxed{V = \frac{1}{3}\pi\left(\frac{R}{H}h\right)^2 h = \frac{1}{3}\pi\left(\frac{R}{H}\right)^2 h^3 \,.}$$

(iv) Calculus: Differentiating and applying the chain rule we find

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{3}\pi\left(\frac{R}{H}\right)^2 3h^2\frac{\mathrm{d}h}{\mathrm{d}t} = \pi\left(\frac{R}{H}\right)^2 h^2\frac{\mathrm{d}h}{\mathrm{d}t} \,.$$

(v) Solve: In this problem $\frac{R}{H} = \frac{1}{6}$, and at the given time, $\frac{\mathrm{d}V}{\mathrm{d}t} = 5$, $h = 5$ so

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{5}{\pi\frac{1}{6^2}5^2} = \frac{36}{5\pi}\frac{\mathrm{m}}{\mathrm{min}} \,.$$

(b) The drain is unclogged and water begins to clear at the rate of $15\text{m}^3/\text{min}$ (but rain is still falling). At what height is the water falling at the rate of 40m/min?

**Solution** We return to the relation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \pi \left(\frac{R}{H}\right)^2 h^2 \frac{\mathrm{d}h}{\mathrm{d}t}.$$

We are now given $\frac{\mathrm{d}V}{\mathrm{d}t} = 5 - 15 = -10\frac{\text{m}^3}{\text{min}}$, $\frac{\mathrm{d}h}{\mathrm{d}t} = -40\frac{\text{m}}{\text{min}}$ so

$$h^2 = \frac{-10}{-40\pi(1/6)^2} = \frac{6^2}{2^2\pi}$$

and

$$h = \frac{3}{\sqrt{\pi}}\text{m}$$

at the given time.

## 2. The Linear Approximation

**Fact.** *For $x$ near $a$ we have $f(x) \approx L(x)$ where*

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$

(1) Use a linear approximation to estimate

(a) $\sqrt{1.2}$

**Solution** Let $f(x) = \sqrt{x}$. We need to approximate $f(1.2)$ so we'll use a linear approximation about $a = 1$. We have $f(a) = f(1) = \sqrt{1} = 1$ and since $f'(x) = \frac{1}{2\sqrt{x}}$ that $f'(a) = f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$. The linear approximation is therefore

$$\sqrt{1.2} \approx 1 + \frac{1}{2}(1.2-1) = 1 + \frac{1}{2}(0.2) = 1.1\,.$$

**Remark (not relevant to solving problem)** To get a better appoximation we can use our approximation $1.1^2 = 1.21 \approx 1.2$ to switch and use $a = 1.21$. We have $f(a) = \sqrt{1.21} = 1.1$ $f'(a) = \frac{1}{2\sqrt{a}} = \frac{1}{2 \cdot 1.1} = \frac{1}{2.2}$ and hence

$$\sqrt{1.2} \approx 1.1 + \frac{1}{2.2}(1.2 - 1.21) = 1.1 - \frac{0.01}{2.2} = 1.1 - \frac{1}{220} \approx 1.095\,.$$

Repeatedly using this idea is known as "Newton's Method".

(b) $(15)^{1/4}$

**Solution** Let $f(x) = x^{1/4}$. We need to approximate $f(15)$. Since $16^{1/4}$ is easy to calculate we'll use a linear approximation about $a = 4$. We have $f(a) = f(16) = 16^{1/4} = 2$ and since $f'(x) = \frac{1}{4}x^{-3/4}$ that

$$f'(a) = \frac{1}{4}(16)^{-3/4} = \frac{1}{4}\left((16)^{1/4}\right)^{-3} = \frac{1}{4}(2)^{-3} = \frac{1}{4 \cdot 8} = \frac{1}{32}\,.$$

The linear approximation is therefore

$$(15)^{1/4} \approx 2 + \frac{1}{32}(15-16) = 2 - \frac{1}{32} = \frac{63}{32}\,.$$

(c) $\log 3$

**Solution 1** Let $f(x) = \log x$. We need to approximate $f(3)$. We know $f(1) = \log 1 = 0$ and $f'(x) = \frac{1}{x}$ so $f'(1) = 1$ so try linear approximation about $a = 1$. Get

$$\log 3 \approx 0 + 1\,(3-1) = 2\,.$$

**Solution 2** The problem was that 3 was too far away from 1. Noticing that $\log 3 = -\log\frac{1}{3}$ let's again approximate about $a = 1$ to get:

$$\log 3 = -\log\frac{1}{3} \approx -\left(0 + 1\left(\frac{1}{3} - 1\right)\right) = \frac{2}{3}\,.$$

This is surely too small ($3 > e$ so $\log 3 > 1$), but better.

**Solution 3** Why not try expanding about $a = e$? We know that $\log e = 1$ and since $(\log x)' = \frac{1}{x}$ that the derivative at $e$ is $\frac{1}{e}$ so

$$\log 3 \approx \log e + \frac{1}{e}(3-e) = 1 + \frac{3}{e} - 1 = \frac{3}{e}\,.$$

This solution is less satisfactory than the first two since we it's not a rational number: it depends on having a precise value for $e$.