Lecture 20: Curve Sketching II

$f(x) = x^{2/3}(x-1)$ (convention. $x^{2/3} = (x^{1/3})^2$, with $x^{1/3}$ defined everywhere)

Domain: $\mathbb{R}$, cts (defined by formula)

Positive if $x>1$, negative if $x<1$ except zero at $x=0$, $x=1$.

No vertical asymptotes, (no horizontal asymptotes)

Derivative: $\frac{2}{3}x^{-1/3}(x-1) + x^{2/3} = \frac{2(x-1)+3x}{3x^{1/3}} = \frac{5x-2}{3x^{1/3}}$

undefined (singularity) at $x=0$,

vanishes (critical number) at $x = 2/5$.

positive on $x<0$, negative on $(0, 2/5)$, positive on $(\frac{2}{5}, \infty)$

(denominator positive for $x>0$, negative for $x<0$ | numerator positive $x > 2/5$, negative $x < 2/5$)

2nd: $f''(x) = \frac{10x+2}{9x^{4/3}}$ vanishes at $-\frac{1}{5}$, undef at $x=0$.

since $x^{4/3} > 0$ for $x \neq 0$, f'' is negative on $(-\infty, -\frac{1}{5})$

positive $(-\frac{1}{5}, 0)$, $(0, \infty)$

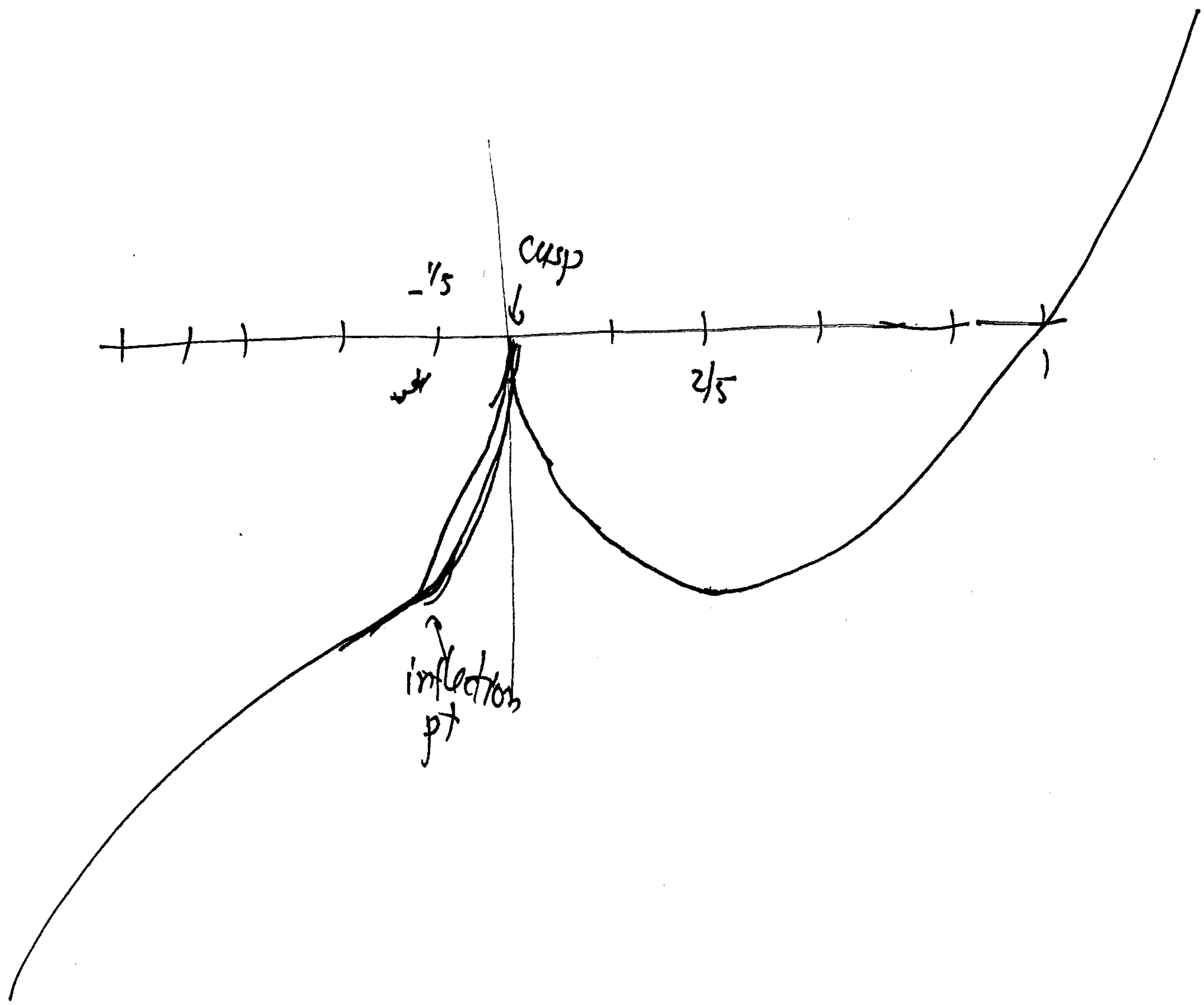
notable x-values: $-\frac{1}{5}$, 0, $2/5$, $x=1$

x	$(-\infty, -\frac{1}{5})$	$-\frac{1}{5}$	$(-\frac{1}{5}, 0)$	0	$(0, 2/5)$	$\frac{2}{5}$	$(\frac{2}{5}, 1)$	1	$(1, \infty)$
f	$-$	$-$	$-$	0	$-$	$-$	$-$	0	$+$
f'	$+$	$+$	$+$	sing max	$-$	0 min	$+$	$+$	$+$
f''	$-$	0	$+$	undef	$+$	$+$	$+$	$+$	$+$

inflection pt

$f(-\frac{1}{5}) = 5^{-2/3}(-\frac{6}{5}) = -\frac{6}{5^{5/3}}$, $f(0) = 0$, $f(\frac{2}{5}) = (\frac{2}{5})^{2/3}(-\frac{3}{5}) = -\frac{2^{2/3} \cdot 3}{5^{5/3}}$

note $\lim_{x\to 0} |f'(x)| = \infty$, so vertical tangent line there


cusp
-1/5
2/5
1
inflection pt

[16] **4.** Let $f(x) = x\sqrt{3-x}$.

(a) (2 marks) Find the domain of $f(x)$.

Answer $(-\infty, 3]$ (or $x \le 3$)

$\{x \in \mathbb{R} \mid x \le 3\}$

f defined where $3 - x \geq 0$

i.e. where $3 \geq x$

(b) (4 marks) Determine the x-coordinates of the local maxima and minima (if any) and intervals where $f(x)$ is increasing or decreasing.

$$f'(x) = \sqrt{3-x} + x \cdot \frac{1}{2\sqrt{3-x}} \cdot (-1) = \frac{2(3-x) - x}{2\sqrt{3-x}} = \frac{6-3x}{2\sqrt{3-x}} = \frac{3}{2} \cdot \frac{2-x}{\sqrt{3-x}}$$

pdt

$f'(x) = 0$ at $x = 2$, undef at $x = 3$

$f' > 0$ if $x < 2$, $f' < 0$ if $2 < x < 3$

so local max at $x = 2$.

(c) (2 marks) Determine intervals where $f(x)$ is concave upwards or downwards, and the x-coordinates of inflection points (if any). You may use, without verifying it, the formula $f''(x) = (3x - 12)(3 - x)^{-3/2}/4$.

if $f''(x) = \frac{3}{4} \cdot \frac{x-4}{(3-x)^{3/2}}$, for $x < 3$ we have $x - 4 < -1 < 0$

$\sqrt{3-x} > 0$

so $f''(x)$ is negative on $(-\infty, 3)$ (undef at $x = 3$)

and f is concave ~~up~~ down

(numerator is >0 on $(4, \infty)$, negative on $(-\infty, 4)$ but domain of f is $(-\infty, 3]$)

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Question 4 continued

(d) (2 marks) There is a point at which the tangent line to the curve $y = f(x)$ is vertical. Find this point.

we have $\lim_{x\to 3^-} f'(x) = -\infty$, so vertical tangent line at $x=3$

$$f'(x) = \frac{3}{2}\,\frac{2-x}{\sqrt{3-x}}, \quad 2-x \xrightarrow[x\to 3]{} -1, \quad \frac{1}{\sqrt{3-x}} \xrightarrow[x\to 3]{} +\infty$$

Answer $x=3$

(e) (2 marks) The graph of $y = f(x)$ has no asymptotes. However, there is a real number a for which $\displaystyle\lim_{x\to-\infty} \frac{f(x)}{|x|^a} = -1$. Find the value of a.

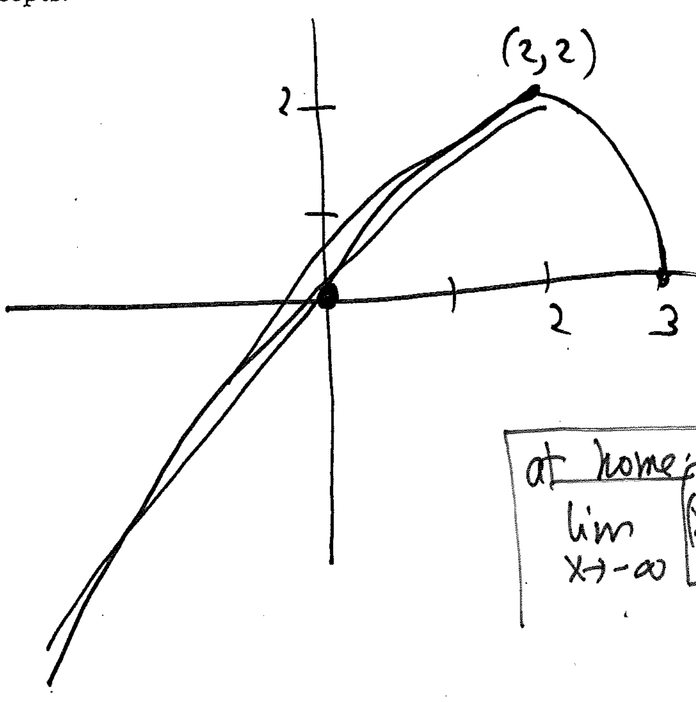
for $|x|$ large, $3-x \sim -x \sim |x|$
so $\sqrt{3-x} \sim \sqrt{|x|}$, $x\sqrt{3-x} \sim -|x|\cdot|x|^{1/2}$
$= -|x|^{3/2}$

Answer $a = 3/2$

so indeed:
$$\lim_{x\to-\infty} \frac{f(x)}{|x|^{3/2}} = \lim_{x\to-\infty} \frac{x}{|x|}\cdot\frac{\sqrt{3-x}}{\sqrt{|x|}} = \lim_{x\to-\infty}\left(\frac{x}{-x}\right)\cdot\sqrt{\frac{3-x}{-x}} = -\lim_{x\to-\infty}\sqrt{1-\frac{3}{x}}$$
$$= -\sqrt{1-0} = -1.$$

(f) (4 marks) Sketch the graph of $y = f(x)$, showing the features given in items (a) to (d) above and giving the (x, y) coordinates for all points occurring above and also all x-intercepts.

$f(3) = 0$
$f(2) = 2\sqrt{3-2} = 2$
$f(0) = 0$



at home:
$$\lim_{x\to-\infty}\left[(x\sqrt{3-x}) - \left(-|x|^{3/2}\right)\right]$$

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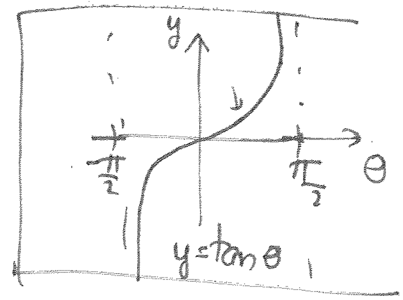
[14] **4.** Let

$$f(x) = \begin{cases} \frac{4}{\pi}\tan^{-1} x, & \text{if } x \geq 1, \\ 2 - x^4, & \text{if } x < 1. \end{cases}$$



[*Note:* Another notation for $\tan^{-1}$ is arctan.]

(a) (3 marks) Show that $f(x)$ is continuous at $x = 1$.

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} \frac{4}{\pi}\arctan x = \underbrace{\frac{4}{\pi}\arctan(1)}_{f(1)} = \frac{4}{\pi}\cdot\frac{\pi}{4} = 1$

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-}(2-x^4) = 2-1^4 = 1$ so $\lim_{x\to1^-} f(x) = f(1) = \lim_{x\to1^+} f(x)$

& f is cts at $x=1$

(b) (1 mark) Determine the equations of any asymptotes (horizontal, vertical or slant).

no vertical asymptotes (f cts on $\mathbb{R}$)

$2-x^4 \underset{x\to-\infty}{\to} -\infty$, but $\lim_{x\to\infty}\frac{4}{\pi}\arctan x = \frac{4}{\pi}\cdot\frac{\pi}{2} = 2$, horizontal asymptote

$y=2$ as $x\to\infty$.

(c) (4 marks) Determine all critical numbers, open intervals where f is increasing or decreasing, and the x-coordinates of all local maxima or local minima (if any).

$f'(x) = -4x^3$ if $x<1$

$f'(x) = \frac{4}{\pi(1+x^2)}$ if $x>1$

$f'(1)$ undefined

so

on left $\lim_{h\to0^-}\frac{f(1+h)-f(1)}{h} = \lim_{h\to0^-}\frac{(2-(1+h)^4)-1}{h} = (-4x^3)\Big|_{x=1} = -4$

$\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1} = \lim_{x\to1^+}\frac{\frac{4}{\pi}\arctan x - 1}{x-1} = \frac{4}{\pi(1+x^2)}\Big|_{x=1} = \frac{2}{\pi} \neq -4$

for $x>1$, $f'(x)>0$

for $0<x<1$, $f'(x)<0$

$f'(0)=0$

for $x<0$, $f'(x)>0$

so local max at $x=0$

local min at $x=1$

Question 4 continues on the next page...

Question 4 continued

(d) (2 marks) Determine open intervals where the graph of f is concave upwards or concave downwards, and the x-coordinates of all inflection points (if any).

$$f''(x) = \begin{cases} -12x^2 & x < 0 \\ -\dfrac{8x}{\pi(1+x^2)^2} & x > 1 \end{cases}$$

so $f''(x)$ is < 0 on $(-\infty, 0)$, $(0, 1)$

$f''(0) = 0 \leftarrow$ not inflection pt)

$f''(x) < 0$ on $x > 1$

no

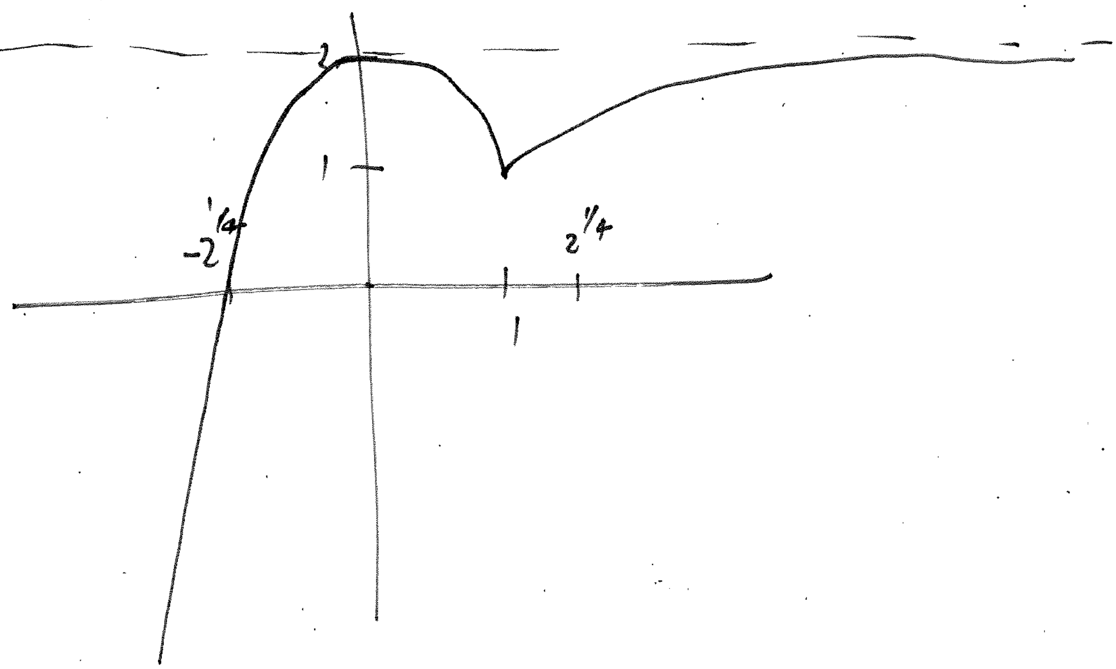
(e) (4 marks) Sketch the curve $y = f(x)$, showing all the features given in items (a) to (d) above and giving the (x, y) coordinates for all points occurring above (if any).

$f(1) = 1$

$f(0) = 2$

~~f(...)~~

$f(\pm 2^{1/4}) = 0$



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