Math 101 – SOLUTIONS TO WORKSHEET 3
THE DEFINITE INTEGRAL

(1) (Sums) Given $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ find
  (a) $\sum_{i=1}^{2n} i$
  (b) $\sum_{i=1}^{n} (2i)$
  Solution: (a) The upper endpoint is $2n$ so the we have $\sum_{i=1}^{2n} i = \frac{(2n)((2n)+1)}{2} = \boxed{n(2n+1)}$.
  (b) $\sum_{i=1}^{n}(2i) = 2\sum_{i=1}^{n} i = \boxed{n(n+1)}$.

(2) (Riemann sums)
  (a) Express the area between the x-axis, the lines $x = 1$ and $x = 4$ and the graph of $f(x) = \cos(x^2)$ as a limit. Use the right-hand rule.
  Solution: We have $a = 1$, $b = 4$, $\Delta x = \frac{b-a}{n} = \frac{3}{n}$, $x_i = a + i\Delta x = 1 + \frac{3i}{n}$ so the area is

$$\boxed{\lim_{n\to\infty} \sum_{i=1}^{n} \cos\left(1 + \frac{3i}{n}\right)^2 \frac{3}{n}\,.}$$

  (b) Express $\lim_{n\to\infty} \frac{1}{2n} \sum_{i=1}^{n} \tan\left(\frac{i}{3n}\right)$ as an integral and as an area.
  Solution: Using $\Delta x = \frac{1}{2n}$ so the points $x_i = \frac{i}{2n}$ range from $x_0 = 0$ to $x_n = \frac{1}{2}$. Since $\tan\left(\frac{i}{3n}\right) = \tan\left(\frac{2}{3}\frac{i}{2n}\right)$ we have $f(x) = \tan(\frac{2}{3}x)$ and the limit is $\int_0^{1/2} \tan(\frac{2}{3}x)\,\mathrm{d}x$, expressing the area between the x-axis, the graph of $f(x) = \tan\left(\frac{2}{3}x\right)$ and the lines $x = 0$ and $x = \frac{1}{2}$.
  Solution: (alternative) Using $\Delta x = \frac{1}{n}$ the points $x_i = \frac{i}{n}$ range from $x_0 = 0$ to $x_n = 1$. Writing the limit as $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}\tan\left(\frac{1}{3}\frac{i}{n}\right) \cdot \frac{1}{n}$ we get $\int_0^1 \frac{1}{2}\tan\left(\frac{1}{3}x\right)\mathrm{d}x$.
  Solution: (alternative) Using $\Delta x = \frac{1}{2n}$ the points $x_i = 1 + \frac{i}{2n}$ range from $x_0 = 1$ to $x_n = \frac{3}{2}$. Writing the limit as $\lim_{n\to\infty} \sum_{i=1}^{n} \tan\left(\frac{2}{3}\left(1 + \frac{i}{2n} - 1\right)\right) \cdot \frac{1}{2n}$ we get $\int_1^{3/2} \tan\left(\frac{2}{3}(x-1)\right)\mathrm{d}x$.

  Remark. For any choice of Δx (proportional to $\frac{1}{n}$) and any choice of a, there is a solution, and they are all correct. The first choice is perhaps the most natural one, but there is no one single answer to this problem. Those who already know about "change of variables" in integrals can see who all the answers are related.

(3) Evaluate
  (a) $\int_0^7 4x\,dx$

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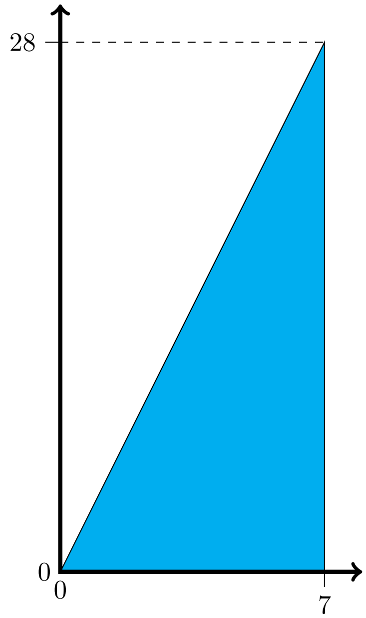
Solution: This is the area of a right-angled triangle with base $[0,7]$ and height 28 so area $\frac{1}{2}\cdot 7\cdot 28 = \boxed{98}$.



(b) $\int_{-1}^{1}\sqrt{1-x^2}dx$

Solution: This is the area of a semicircle of radius 1, so $\boxed{\dfrac{1}{2}\pi}$.

(c) $\int_{-2}^{2}(3+x)dx$

Solution: We use linearity: $\int_{-2}^{2}(3+x)dx = \int_{-2}^{2}3dx + \int_{-2}^{2}xdx$. The first part is the area of a rectangle of width 4 and height 3, which is 12. In the second part, $\int_{-2}^{2}xdx = \int_{-2}^{0}xdx + \int_{0}^{2}xdx$ so we we need add the *signed* areas of two otherwise identical triangles, one above and one below the axis. The signed areas cancel so $\int_{-2}^{2}xdx = 0$ and

$$\boxed{\int_{-2}^{2}(3+x)dx = 12\,.}$$