Math 101 – SOLUTIONS TO WORKSHEET 11
INTEGRATION BY PARTS

(1) Evaluate the integrals

(a) $\int xe^x \, \mathrm{d}x$

Solution: Let $u = x$, $\mathrm{d}v = e^x \, \mathrm{d}x$ so that $v = \int e^x \, \mathrm{d}x = e^x$. Then $\mathrm{d}u = \mathrm{d}x$ so that

$$\int xe^x \, \mathrm{d}x = \int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u = xe^x - \int e^x \, \mathrm{d}x = xe^x - e^x + C = (x-1)e^x + C\,.$$

(b) (Final, 2014) $\int x \log x \, \mathrm{d}x$

Solution: This time, let $u = \log x$, $\mathrm{d}v = x \, \mathrm{d}x$ so that $v = \frac{1}{2}x^2$ and $\mathrm{d}u = \frac{1}{x} \, \mathrm{d}x$. Integrating by parts, we get:

$$\begin{aligned} \int x \log x \, \mathrm{d}x &= \frac{1}{2}x^2 \log x - \int \frac{1}{2}x^2 \cdot \frac{1}{x} \, \mathrm{d}x \\ &= \frac{1}{2}x^2 \log x - \frac{1}{2} \int x \, \mathrm{d}x \\ &= \frac{1}{2}x^2 \log x - \frac{1}{4}x^2 + C\,. \end{aligned}$$

(c) $\int x^2 \cos x \, \mathrm{d}x$

Solution: We integrate by parts twice, differentiating the term of the form x^k

$$\begin{aligned} \int x^2 \cos x \, \mathrm{d}x &= x^2 \sin x - \int (2x) \sin x \, \mathrm{d}x \\ &= x^2 \sin x - \left[(-2x \cos x) - \int (2)(-\cos x) \, \mathrm{d}x \right] \\ &= x^2 \sin x + 2x \cos x - 2 \int \cos x \, \mathrm{d}x \\ &= x^2 \sin x + 2x \cos x - 2 \sin x + C\,. \end{aligned}$$

(d) $\int \log x \, \mathrm{d}x$

Solution: Let $u = \log x$, $\mathrm{d}v = 1 \, \mathrm{d}x = \mathrm{d}x$ so that $v = x$, $\mathrm{d}u = \frac{1}{x} \, \mathrm{d}x$. Integrating by parts, we get:

$$\begin{aligned} \int \log x \, \mathrm{d}x &= x \log x - \int x \cdot \frac{1}{x} \, \mathrm{d}x \\ &= x \log x - \int \mathrm{d}x \\ &= x \log x - x + C\,. \end{aligned}$$

(e) (Final, 2013) $\int_0^1 \arctan x \, \mathrm{d}x =$

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Solution: Let $u = \arctan x$, $\mathrm{d}v = 1\,\mathrm{d}x = \mathrm{d}x$ so that $v = x$, $\mathrm{d}u = \frac{\mathrm{d}x}{1+x^2}$. Integrating by parts, we get:

$$\begin{aligned}
\int \arctan x\,\mathrm{d}x &= x\arctan x - \int \frac{x\,\mathrm{d}x}{1+x^2} \\
&\overset{w=1+x^2}{=} x\arctan x - \int \frac{\frac{1}{2}\,\mathrm{d}w}{w} \qquad (\mathrm{d}w = 2x\,\mathrm{d}x) \\
&= x\arctan x - \frac{1}{2}\log|w| + C \\
&= x\arctan x - \frac{1}{2}\log\left(1+x^2\right) + C\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^1 \arctan x\,\mathrm{d}x &= \left[x\arctan x - \frac{1}{2}\log\left(1+x^2\right) + C\right] = \arctan 1 - \frac{1}{2}\log 2 - \frac{1}{2}\log 1 \\
&= \frac{\pi}{4} - \frac{1}{2}\log 2\,.
\end{aligned}$$

(2) Now let's play with out toolkit

(a) Evaluate $\int \frac{\log x}{x}\,\mathrm{d}x$

Solution: Let $u = \log x$ so that $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$. We then have

$$\int \frac{\log x}{x}\,\mathrm{d}x = \int u\,\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}\log^2 x + C\,.$$

(b) Evaluate $\int \frac{\log x}{x^2}\,\mathrm{d}x$

Solution: This time we integrate by parts:

$$\begin{aligned}
\int \frac{\log x}{x^2}\,\mathrm{d}x &= \left(-\frac{1}{x}\right)\log x - \int \left(-\frac{1}{x}\right)\frac{1}{x}\,\mathrm{d}x \\
&= -\frac{\log x}{x} + \int \frac{1}{x^2}\,\mathrm{d}x \\
&= -\frac{\log x}{x} - \frac{1}{x} + C \\
&= -\frac{\log x + 1}{x} + C
\end{aligned}$$

(c) (Final, 2010) Let $g(x) = \int_0^1 \left(xe^t - t\right)^2 \mathrm{d}t$. Find the minimum value of $g(x)$.

Solution: We have

$$\begin{aligned}
g(x) &= \int_0^1 \left(x^2e^{2t} - 2xte^t + t^2\right)\mathrm{d}t = \\
&= x^2\int_0^1 e^{2t}\,\mathrm{d}t + \int_0^1 t^2 - 2x\int_0^1 te^t\,\mathrm{d}t \\
&= x^2\left[\frac{1}{2}e^{2t}\right]_{t=0}^{t=1} + \left[\frac{t^3}{3}\right]_{t=0}^{t=1} - 2x\left[te^t\right]_{t=0}^{t=1} + 2x\int_0^1 e^t\,\mathrm{d}t \\
&= \frac{e^2-1}{2}x^2 + \frac{1}{3} - 2ex + 2x\left[e^t\right]_{t=0}^{t=1} \\
&= \frac{e^2-1}{2}x^2 + 2(e-1)x - 2ex + \frac{1}{3} \\
&= \frac{e^2-1}{2}x^2 - 2x + \frac{1}{3}\,.
\end{aligned}$$

This is a concave-up parabola so has a unique minimum. We have $g'(x) = \big(e^2-1\big)\,x - 2$ and the minimum is where $g'(x) = 0$ that is where

$$x = \frac{2}{e^2-1}\,.$$

(d) Evaluate $\int x^3 \log(x^2+1)\,\mathrm{d}x$

Solution: We "peel of" and x, substituting $u = x^2+1$ so that $x^2 = u-1$ and $\mathrm{d}u = 2x\,\mathrm{d}x$ so that

$$\begin{aligned}
\int x^3 \log(x^2+1)\,\mathrm{d}x &= \frac{1}{2}\int x^2\log(x^2+1)\cdot 2x\,\mathrm{d}x \\
&= \frac{1}{2}\int (u-1)\log u\,\mathrm{d}u \\
&= \frac{1}{2}\left(\frac{1}{2}u^2-u\right)\log u - \frac{1}{2}\int\left(\frac{1}{2}u^2-u\right)\frac{1}{u}\,\mathrm{d}u \\
&= \frac{1}{4}\left(u^2-2u\right)\log u - \frac{1}{4}\int (u-2)\,\mathrm{d}u \\
&= \frac{1}{4}\left(u^2-2u\right)\log u - \frac{1}{8}u^2 + \frac{1}{2}u + C \\
&= \frac{1}{4}\left((x^2+1)^2 - 2(x^2+1)\right)\log\left(x^2+1\right) - \frac{1}{8}\left(x^2+1\right)\left(x^2+1-4\right) + C \\
&= \frac{1}{4}\left(x^4-1\right)\log\left(x^2+1\right) - \frac{1}{8}\left(x^4-2x^2\right) + C\,.
\end{aligned}$$

Solution: We start by integrating by parts, using $u = \log(x^2+1)$, $\mathrm{d}v = x^3\,\mathrm{d}x$ getting:

$$\begin{aligned}
\int x^3 \log(x^2+1)\,\mathrm{d}x &= \frac{1}{4}x^4\log(x^2+1) - \frac{1}{4}\int x^4\frac{2x}{x^2+1}\,\mathrm{d}x \\
&= \frac{1}{4}x^4\log\left(x^2+1\right) - \frac{1}{2}\int \frac{x^5}{x^2+1}\,\mathrm{d}x \\
&= \frac{1}{4}x^4\log\left(x^2+1\right) - \frac{1}{2}\int\left(\frac{x^5+x^3}{x^2+1} - \frac{x^3+x}{x^2+1} + \frac{x}{x^2+1}\right)\mathrm{d}x \\
&= \frac{1}{4}x^4\log\left(x^2+1\right) - \frac{1}{2}\int\left(x^3 - x + \frac{x}{x^2+1}\right)\mathrm{d}x \\
&= \frac{1}{4}x^4\log\left(x^2+1\right) - \frac{1}{8}x^4 + \frac{1}{4}x^2 - \frac{1}{4}\int\frac{\mathrm{d}u}{u} \qquad u = x^2+1, \mathrm{d}u = 2x\,\mathrm{d}x \\
&= \frac{1}{4}x^4\log\left(x^2+1\right) - \frac{1}{8}x^4 + \frac{1}{4}x^2 - \frac{1}{4}\log u + C \\
&= \frac{1}{4}\left(x^4-1\right)\log\left(x^2+1\right) - \frac{1}{8}\left(x^4-2x^2\right) + C\,.
\end{aligned}$$