Math 101 – SOLUTIONS TO WORKSHEET 17
APPROXIMATE INTEGRATION

1. Approximate integration

(1) Let $f(x) = \sin(x^2)$. Estimate $\int_0^1 f(x)\,\mathrm{d}x$ using the trapezoid rule, the midpoint rule, and Simpson's rule, with $n = 4$ in all cases. You may leave your answers in calculator-ready form.

Solution: With $n = 4$ we have $\Delta x = \frac{1}{4}$ and the points $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$, so the approximations are:

$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x &\approx \frac{1}{8}\left(\sin(0^2) + 2\sin\left(\left(\frac{1}{4}\right)^2\right) + 2\sin\left(\left(\frac{1}{2}\right)^2\right) + 2\sin\left(\left(\frac{3}{4}\right)^2\right) + \sin\left(1^2\right)\right) \\ &= \frac{1}{8}\left(2\sin\left(\frac{1}{16}\right) + 2\sin\left(\frac{1}{4}\right) + 2\sin\left(\frac{9}{16}\right) + \sin(1)\right),\end{aligned}$$

$$\int_0^1 f(x)\,\mathrm{d}x \approx \frac{1}{4}\left(\sin\left(\left(\frac{1}{8}\right)^2\right) + \sin\left(\frac{9}{64}\right) + \sin\left(\frac{25}{64}\right) + \sin\left(\frac{49}{64}\right)\right)$$

and

$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x &\approx \frac{1}{12}\left(\sin(0) + 4\sin\left(\frac{1}{16}\right) + 2\sin\left(\frac{1}{4}\right) + 4\sin\left(\frac{9}{16}\right) + \sin(1)\right) \\ &= \frac{1}{12}\left(4\sin\left(\frac{1}{16}\right) + 2\sin\left(\frac{1}{4}\right) + 4\sin\left(\frac{9}{16}\right) + \sin(1)\right).\end{aligned}$$

(2) (Final 2009) Give the Simpson's rule approximation to $\int_0^2 \sin(e^x)\,\mathrm{d}x$ using 4 equal subintervals.

Solution: Here $\Delta x = \frac{2}{4} = \frac{1}{2}$, the points are $0, \frac{1}{2}, 1, \frac{3}{2}, 2$ and so the approximation is

$$\frac{1}{6}\left(\sin\left(e^0\right) + 4\sin\left(e^{1/2}\right) + 2\sin\left(e^1\right) + 4\sin\left(e^{3/2}\right) + \sin\left(e^2\right)\right)$$

which is

$$\frac{1}{6}\left(\sin(1) + 4\sin\left(e^{1/2}\right) + 2\sin(e) + 4\sin\left(e^{3/2}\right) + \sin\left(e^2\right)\right).$$

(3) (Final 2012) Let $I = \int_1^2 \frac{1}{x}\,\mathrm{d}x$.

(a) Write down Simpson's rule approximation for I using 4 points (call it S_4)

Solution: $S_4 = \frac{1}{12}\left(\frac{1}{1} + 4\frac{1}{5/4} + 2\frac{1}{3/2} + 4\frac{1}{7/4} + \frac{1}{2}\right)$.

<u>It was not required to do the arithmetic</u>, but for the record we note (since $210 = 2 \cdot 3 \cdot 5 \cdot 7$):

$$\begin{aligned}S_4 &= \frac{1}{12}\left(1 + \frac{16}{5} + \frac{4}{3} + \frac{16}{7} + \frac{1}{2}\right) \\ &= \frac{1}{12} \cdot \frac{210 + 42 \cdot 16 + 70 \cdot 3 + 30 \cdot 16 + 105}{210} \\ &= \frac{1677}{2520}.\end{aligned}$$

(b) Without computing I, find an upper bound for $|I - S_4|$. You may use the fact that if $\left|f^{(4)}(x)\right| \le K$ on $[a, b]$ then the error in the approximation with n points has magnitude at most $K(b-a)^5/180n^4$.

Date: 12/2/2016, Worksheet by Lior Silberman.

Solution: We have $f'(x) = -\frac{1}{x^2}$, $f^{(2)}(x) = \frac{2}{x^3}$, $f^{(3)}(x) = -\frac{6}{x^4}$ and $f^{(4)}(x) = \frac{24}{x^5}$. On the interval $[1, 2]$, the function $\frac{24}{x^5}$ is decreasing so $\left|f^{(4)}(x)\right| \le \frac{24}{1} = 24$. It follows that the error is at most

$$\frac{24(2-1)^5}{180 \cdot 4^4} = \frac{24}{180 \cdot 256} = \frac{1}{60 \cdot 32} = \frac{1}{1960}\,.$$

(4) (Final 2008) Let $I = \int_0^1 \cos(x^2)\,\mathrm{d}x$. It can be shown that the fourth derivative of $\cos(x^2)$ has absolute value at most 60 on $[0, 1]$. Find n such the Simpson's rule approximation to I using n points has error less than or equal to 0.001. You may use that that if $\left|f^{(4)}(t)\right| \le K$ for $a \le t \le b$ then error in using Simpson's rule to approximate $\int_a^b f(x)\,\mathrm{d}x$ has absolute value less than or equal to $K(b-a)^5/180n^4$.

Solution: For $f(x) = \cos(x^2)$ we are given that $\left|f^{(4)}(x)\right| \le 60$ for $1 \le x \le 2$, so we need n such that $\frac{60 \cdot (1-0)^5}{180n^4} \le \frac{1}{1000}$, that is

$$\frac{1}{3n^4} \le \frac{1}{1000}$$

which is the same as

$$n^4 \ge \frac{1000}{3}\,.$$

Now for $n = 6$ we have $6^4 = 36 \cdot 36 \ge 30 \cdot 30 = 900 > \frac{1000}{3}$ so $n = 6$ suffices.

(5) Let $I = \int_4^6 \sin(\sqrt{x})\,\mathrm{d}x$. Find n such that estimating I using the midpoint rule and n points will have an error of at most $\frac{1}{3000}$. You may use that the absolute error in estimating $\int_a^b f(x)\,\mathrm{d}x$ using the midpoint rule and n points is at most $K(b-a)^3/24n^2$ where $\left|f^{(4)}(x)\right| \le K$ for $a \le x \le b$.

Solution: Let $f(x) = \sin(\sqrt{x})$. Then $f'(x) = \frac{1}{2\sqrt{x}}\cos(\sqrt{x})$ so $f''(x) = -\frac{1}{4x^{3/2}}\cos(\sqrt{x}) - \frac{1}{4x}\sin(\sqrt{x})$. For $4 \le x \le 6$ we have $\frac{1}{4x^{3/2}} \le \frac{1}{4 \cdot 4^{3/2}} = \frac{1}{32}$ ($\frac{1}{x^{3/2}}$ is decreasing on this interval) and $\frac{1}{4x} \le \frac{1}{4 \cdot 4} = \frac{1}{16}$ (for the same reason). Since $|\cos(\sqrt{x})|, |\sin(\sqrt{x})| \le 1$ for all x, we have

$$\left|f^{(2)}(x)\right| \le \frac{1}{32} + \frac{1}{16} = \frac{3}{32} \le \frac{3}{30} = \frac{1}{10}$$

for all $4 \le x \le 6$. It follows that the error in the approximation is at most

$$\frac{1}{10} \cdot \frac{(6-4)^3}{24 \cdot n^2} = \frac{8}{240n^2} = \frac{1}{30n^2}\,.$$

For $n = 10$ the error would be at most $\frac{1}{30 \cdot 100} = \frac{1}{3000}$ so that is enough.