# Math 101 – SOLUTIONS TO WORKSHEET 25
# THE INTEGRAL TEST

## 1. Review of improper integrals

(1) Show that $\int_2^\infty \frac{\mathrm{d}x}{x}$ diverges.

**Solution:** $\int_2^T \frac{\mathrm{d}x}{x} = \log T - \log 2 \xrightarrow[T\to\infty]{} \infty$

(2) Show that $\int_2^\infty \frac{\mathrm{d}x}{x^3+5}$ converges.

**Solution:** For $x > 0$ we have $x^3+5 > x^3 > 0$ so $0 < \frac{1}{x^3+5} < \frac{1}{x^3}$ and $\int_2^T \frac{\mathrm{d}x}{x^3} = \frac{1}{2}\left(\frac{1}{2^2} - \frac{1}{T^2}\right) \xrightarrow[T\to\infty]{} \frac{1}{8}$ so the integral converges by the comparison test.

(3) Evaluate $\int_0^\infty xe^{-x}\,\mathrm{d}x$.

**Solution:** We integrate by parts:

$$\begin{aligned}\int_0^T xe^{-x}\,\mathrm{d}x &= \left[-xe^{-x}\right]_0^T - \int_0^T \left(-e^{-x}\right)\mathrm{d}x = -Te^{-T} + \left[e^{-x}\right]_0^T = 1 - Te^{-T} - e^{-T} \\ &= 1 - \frac{T+1}{e^T}\,.\end{aligned}$$

Now as $T \to \infty$ by l'Hôpital,

$$\lim_{T\to\infty} \frac{T+1}{e^T} = \lim_{T\to\infty} \frac{1}{e^T} = 0$$

so

$$\int_0^\infty xe^{-x}\,\mathrm{d}x = \lim_{T\to\infty} \int_0^T xe^{-x}\,\mathrm{d}x = 1\,.$$

## 2. Applying the integral test

(4) Decide if each series converges or diverges

(a) $\sum_{n=1}^\infty \frac{1}{n^p}$ (your answer will depend on $p$!)

**Solution:** For $p \le 0$, $\frac{1}{n^p}$ does not decay to zero and the series diverges by the divergence test. Let $f(x) = \frac{1}{x^p}$ so that the series is $\sum_{n=1}^\infty f(n)$. The function $f$ is **decreasing** and **positive**, so by the integral test the series converges iff $\int_1^\infty f(x)\,\mathrm{d}x$ converges, that is iff $p > 1$ by the $p$-test.

(b) $\sum_{n=1}^\infty \frac{n}{e^n}$

**Solution:** Let $f(x) = xe^{-x}$, so that the series is $\sum_{n=1}^\infty f(n)$. Then $f(x) > 0$ for all $x$. Also, we have $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$ which is negative for $x > 1$ so $f$ is **eventually decreasing**. We know that $\int_0^\infty xe^{-x}\,\mathrm{d}x$ converges (see problem 3) so by the integral test our series converges as well.

(c) (Final 2014) $\sum_{n=2}^\infty \frac{1}{n(\log n)^p}$ (your answer will depend on $p$!)

**Solution:** Suppose $p > 0$ (if $p \le 0$ compare with $\sum_{n=1}^\infty \frac{1}{n}$ – see next lecture). Let $f(x) = \frac{1}{x(\log x)^p}$ which is clearly positive and decreasing. so that the series is $\sum_{n=1}^\infty f(n)$. The function $f$ is clearly both **positive** and **decreasing**, so by the integral test the series converges iff $\int_2^\infty f(x)\,\mathrm{d}x$ converges. We consider

$$\int_2^\infty \frac{\mathrm{d}x}{x(\log x)^p}\,.$$

*Date*: 9/3/2016, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Susbtituting $u = \log x$ we have $\frac{\mathrm{d}x}{x} = \mathrm{d}u$ and $u \to \infty$as $x \to \infty$ so we have

$$\int_2^\infty \frac{\mathrm{d}x}{x(\log x)^p} = \int_2^\infty \frac{\mathrm{d}u}{u^p}$$

which converges when $p > 1$ and diverges otherwise. By the integral test the same holds for our series.

(d) $\sum_{n=1}^\infty \frac{1}{n^2+1}$

**Solution:** Let $f(x) = \frac{1}{1+x^2}$ which is clearly **positive** and **decreasing**. By the integral test the series $\sum_{n=1}^\infty f(n)$ converges iff the integral $\int_1^\infty \frac{\mathrm{d}x}{1+x^2}$ does. But

$$\int_1^\infty \frac{\mathrm{d}x}{1+x^2} = \lim_{T\to\infty} (\arctan(T) - \arctan(1)) = \lim_{T\to\infty} \arctan(T) - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4},$$

so the integtral and the series converge.

**Solution:** Let $f(x) = \frac{1}{1+x^2}$ which is clearly **positive** and **decreasing**. By the integral test the series $\sum_{n=1}^\infty f(n)$ converges iff the integral $\int_0^\infty \frac{\mathrm{d}x}{1+x^2}$ does. Converges does not depend on the starting point so we consider $\int_1^\infty \frac{1}{1+x^2}\,\mathrm{d}x$. Now $\frac{1}{1+x^2} < \frac{1}{x^2}$ and $\int_1^\infty \frac{\mathrm{d}x}{x^2}$ converges by the $p$-test $(2 > 1)$ so $\int_1^\infty \frac{\mathrm{d}x}{1+x^2}$ converges by the comparison test, and $\sum_{n=1}^\infty \frac{1}{n^2+1}$ converges by the integral test.

(5) The integral $\int_2^\infty \frac{x+\sin x}{1+x^2}\,\mathrm{d}x$ diverges. Why can't we use the integral test to conlcude that $\sum_{n=2}^\infty \frac{n+\sin n}{1+n^2}$ diverges as well?

**Solution:** The function $f(x) = \frac{x+\sin x}{1+x^2}$ isn't monotone:

$$\begin{aligned} f'(x) &= \frac{(1+\cos x)(1+x^2) - 2x(x+\sin x)}{(1+x^2)^2} \\ &= \frac{(1+\cos x - 2)\,x^2 - 2x\sin x + 1 + \cos x}{(1+x^2)^2} \\ &= \frac{(\cos x - 1)x^2 - 2x\sin x + 1 + \cos x}{(1+x)^2}\,. \end{aligned}$$

In particular, if $x = 2\pi k$ $(k \in \mathbb{Z})$ then $\cos x = 1$, $\sin x = 0$ and

$$f'(x) = \frac{2}{(1+x^2)} > 0\,.$$

We'll later show that this series diverges anyway.

## 3. Tail estimates (not examinable)

(6) Consider the series $\sum_{n=1}^\infty \frac{1}{n^2}$

(a) Show that $\sum_{n=N+1}^\infty \frac{1}{n^2} \leq \frac{1}{N}$.

**Solution:** The fucntion $f(x) = \frac{1}{x^2}$ is decreasing and positive. By the integral test, $\sum_{n=N+1}^\infty f(n) \leq \int_N^\infty f(x)\,\mathrm{d}x = \left[-\frac{1}{x}\right]_N^\infty = \frac{1}{N}$.

(b) How many terms to we need to include to get an answer accurate to $10^{-5}$?

**Solution:** We have $\sum_{n=1}^\infty \frac{1}{n^2} = \sum_{n=1}^N \frac{1}{n^2} + \sum_{n=N+1}^\infty \frac{1}{n^2}$. If $N = 10^5$ we see that

$$0 \leq \sum_{n=1}^\infty \frac{1}{n^2} - \sum_{n=1}^{10^5} \frac{1}{n^2} \leq 10^{-5}\,.$$

(7) (The harmonic series)

(a) Show that $\sum_{n=1}^N \frac{1}{n} \geq \log(N+1)$

**Solution:** $\sum_{n=1}^N \frac{1}{n} \geq \int_1^{N+1} \frac{\mathrm{d}x}{x} = \log(N+1)$.

(b) Show that $\sum_{n=1}^N \frac{1}{n} \leq (1 - \log 2) + \log(N+1)$

**Solution:** $\sum_{n=1}^N \frac{1}{n} \leq 1 + \int_2^{N+1} \frac{\mathrm{d}x}{x} = 1 + \log(N+1) - \log 2$.