Math 101 – SOLUTIONS TO WORKSHEET 26
THE COMPARISON TEST

1. Comparison by Massaging

(1) Determine, with explanation, whether the following series converge or diverge.

(a) (Final 2014) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

Solution: For $n \geq 1$ we have $n^2+1 \leq n^2+n^2 = 2n^2$ so that $\frac{1}{\sqrt{n^2+1}} \geq \frac{1}{\sqrt{2n^2}} = \frac{1}{\sqrt{2}}\frac{1}{n}$. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-test with $p = 1 \leq 1$) so by the comparison test the given series diverges as well.

(b) (Final 2013, variant) $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \frac{1}{121} + \cdots$

Solution: The series is $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$ and has positive terms. The nth odd number is $2n-1$ so the series is

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\,.$$

For $n \geq 1$, $2n-1 \geq 2n-n = n$ so $\frac{1}{(2n-1)^2} \leq \frac{1}{n^2}$. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test ($p = 2 > 1$) so by the comparison test our series converges too.

(c) (Final 2013) $\sum_{n=1}^{\infty} \frac{n+\sin n}{1+n^2}$

Solution: For $n \geq 2$ we have $n + \sin n \geq n-1 \geq n - \frac{n}{2}$ and $1+n^2 \leq 2n^2$ so that for $n \geq 2$ we have $\frac{n+\sin n}{n^2+1} \geq \frac{n/2}{2n^2} = \frac{1}{4n}$. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-test with $p = 1$) so by the comparison test the given series diverges as well.

(d) $1 + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{4^3} + \frac{1}{5^2} + \frac{1}{6^3} + \frac{1}{7^2} + \cdots$

Solution: Let a_n be the nth term of the series (which is positive) so that $a_n = \begin{cases} \frac{1}{n^2} & n \text{ odd} \\ \frac{1}{n^3} & n \text{ even} \end{cases}$.

For $n \geq 1$ we have $\frac{1}{n^3} \leq \frac{1}{n^2}$ so $a_n \leq \frac{1}{n^2}$ in any case. Now $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test ($p = 2 > 1$) so by the comparison test the series $\sum_{n=1}^{\infty} a_n$ converges as well.

2. Limit comparison test

(2) Determine, with explanation, whether the following series converge or diverge.

(a) (Final 2014) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

Solution: We have $\lim_{n\to\infty} \frac{1}{n}\Big/\frac{1}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n} = \lim_{n\to\infty} \sqrt{1+\frac{1}{n^2}} = 1$. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-test with $p = 1$) so by the limit comparison test our series dinverges as well.

(b) (Final 2013, variant) $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \frac{1}{121} + \cdots$

Solution: The series is $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$ and has positive terms. The nth odd number is $2n-1$ so the series is

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\,.$$

Now $\lim_{n\to\infty} \frac{1}{n^2}\Big/\frac{1}{(2n-1)^2} = \lim_{n\to\infty} \left(\frac{2n-1}{n}\right)^2 = \lim_{n\to\infty} \left(2 - \frac{1}{n}\right)^2 = 4$. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test ($p = 2 > 1$) so by the limit comparison test our series converges too.

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(c) (Final 2013) $\sum_{n=1}^{\infty} \frac{n+\sin n}{1+n^2}$
Solution: We have

$$\lim_{n\to\infty} \frac{n+\sin n}{n^2+1} \Big/ \frac{1}{n} = \lim_{n\to\infty} \frac{1+\frac{\sin n}{n^2}}{1+\frac{1}{n^2}} = \frac{1+\lim_{n\to\infty}\frac{\sin n}{n^2}}{1+\lim_{n\to\infty}\frac{1}{n^2}}$$

Now $\lim_{n\to\infty} \frac{1}{n^2} = 0$. Since $-1 \le \sin n \le 1$, we have $-\frac{1}{n^2} \le \frac{\sin n}{n^2} \le \frac{1}{n^2}$ and $\lim_{n\to\infty}\left(-\frac{1}{n^2}\right) = -\lim_{n\to\infty}\frac{1}{n^2} = 0$ also so by the squeeze theorem, $\lim_{n\to\infty}\frac{\sin n}{n^2} = 0$. It follows that

$$\lim_{n\to\infty} \frac{n+\sin n}{n^2+1} \Big/ \frac{1}{n} = \frac{1+0}{1+0} = 1\,.$$

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-test with $p=1$) so by the limit comparison test the given series diverges as well.