Math 101 – SOLUTIONS TO WORKSHEET 28
ABSOLUTE CONVERGENCE

1. More Tail Estimates

(1) It is known that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \log 2$. How many terms are needed for the error to be less than 0.01?

Solution: The series is alternating, so the error in approximating its sum by a partial sum is less than the first ommitted term. Taking the first 99 terms, this means that

$$\left|\log 2 - \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{99}\right)\right| \leq \frac{1}{100}$$

as desired.

(2) It is known that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}$. How many terms are needed for the error to be less than 0.001?

Solution: Again the series is alternating. The magnitude of the nth term is $\frac{1}{2n-1}$ so taking the first 500 terms we get that

$$\left|\frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots - \frac{1}{999}\right)\right| \leq \frac{1}{1001} < \frac{1}{1000}\,.$$

2. Convergence

(3) Which of the following converges:

$$\square \left\{\frac{1}{\sqrt{n}}\right\}_{n=1}^{\infty} \quad \square \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad \square \left\{\frac{(-1)^n}{\sqrt{n}}\right\}_{n=1}^{\infty} \quad \square \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Solution: $\lim_{n\to 1} \frac{1}{\sqrt{n}} = 0$, so also $\lim_{n\to\infty} \frac{-1}{\sqrt{n}} = 0$, and by the squeeze theorem $\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n}} = 0$, so both sequences *converge*. The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} < 1$ so it *diverges* while the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ *converges* by the alternating series test.

(4) Place checkmarks

	Converges		Diverges
	Absolutely	Conditionally	
$\sum_{n=1}^{\infty}(-1)^n$			
$\sum_{n=1}^{\infty} \frac{1}{n^2}$			
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$			
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$			
$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$			
$\sum_{n=1}^{\infty} \frac{\sin n}{n}$			

3. Ratio test

(5) Decide whether the following series converge:

(a) $\sum_{n=0}^{\infty} \frac{n}{2^n}$

Solution: We have $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2^{n+1}} \Big/ \frac{n}{2^n} = \frac{n+1}{n} \cdot \frac{2^n}{2^{n+1}} = \frac{1}{2}\left(1 + \frac{1}{n}\right) \xrightarrow[n\to\infty]{} \frac{1}{2} < 1$ so the series converges by the ratio test.

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(b) $\sum_{n=0}^{\infty} \frac{n!}{2^n}$

Solution: We have $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{2^{n+1}} \Big/ \frac{n!}{2^n} = \frac{(n+1)!}{n!} \cdot \frac{2^n}{2^{n+1}} = \frac{n+1}{2} \xrightarrow[n\to\infty]{} \infty > 1$ so the series diverges by the ratio test.

(c) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

Solution: We have $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{n+1} \xrightarrow[n\to\infty]{} 0 < 1$ so the series converges by the ratio test.

(d) For which values of x does $\sum_{n=0}^{\infty} nx^n$ converge?

Solution: Let $a_n = nx^n$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)\,|x|^{n+1}}{n\,|x|^n} = \left(1 + \frac{1}{n}\right)|x| \xrightarrow[n\to\infty]{} |x| \ .$$

By the ratio test, the series *converges* if$|x| < 1$ and *diverges* if $|x| > 1$. If $|x| = 1$ then $|a_n| = n\,|x|^n = n \xrightarrow[n\to\infty]{} \infty$ so the series *diverges* by the divergence test. We conclude that the series converges exactly when $|x| < 1$, that is for $x \in (-1, 1)$.