

Midterm Exam Duration: 80 minutes*This test has 6 questions on 10 pages, for a total of 60 points.*

- Do not turn this page over. You will have 80 minutes for the exam (between 14:00-15:20)
- This is a closed-book examination: no books, notes or electronic devices of any kind.
- You must justify all answers, regardless of the "operative word". Write in complete English sentences; proofs must be clear and concise.
- If you are using a result from the textbook, the lectures or the problem sets, state it properly.
- This exam is printed double-sided with the last two pages blank.

First Name: _____ Last Name: _____

Student-No: _____ Signature: _____

Question:	1	2	3	4	5	6	Total
Points:	15	10	5	10	10	10	60
Score:							

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (iii) purposely viewing the written papers of other examination candidates;
 - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1. Definition

5 marks

(a) Let G be a group. Define a *subgroup* H of G .

6 marks

(b) Let G be a group and let $\{H_i\}_{i=1}^k$ be subgroups of G . Show that $\bigcap_{i=1}^k H_i$ is a subgroup of G .

4 marks

(c) Let G, H be groups and let $f \in \text{Hom}(G, H)$. Show that the image $\text{Im}(f)$ is a subgroup of H .

2. Calculation

5 marks

(a) Solve the system of equations in $\mathbb{Z}/8\mathbb{Z}$:

$$\begin{cases} [5]x + [3]y = [2] \\ [7]x - [2]y = [3] \end{cases}$$

5 marks

(b) Find the cycle decomposition of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 7 & 3 & 1 & 8 & 9 & 6 & 4 \end{pmatrix}$.

No justification is needed for this problem.

3. **Groups.** In each case, decide if the assertion is true. If so, prove the assertion. If not, show where why definition fails to hold.

2 marks

(a) Let $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$ and let \cdot be matrix multiplication. Then (N, \cdot) is a group.

3 marks

(b) Let $X = \mathbb{R} \setminus \{0\}$ be the set of non-zero reals, and let $f: X \rightarrow X$ be the map $f(x) = 2x$. Let \cdot be the usual multiplication of real numbers. Then there is an operation \star on X such that $f: (X, \cdot) \rightarrow (X, \star)$ is a group isomorphism.

4. Groups and Permutations

5 marks

- (a) Let G be a finite group of order n , and let $g \in G$. Without using Lagrange's Theorem, show that g has finite order.

5 marks

- (b) By Lagrange's Theorem, the order d of g divides n . Define a permutation of the set G by $\sigma(x) = gx$ (you don't need to show that this is a permutation). Find the cycle structure of this permutation, that is number of cycles of each length.

5. Commutativity.

Fix a group G .

5 marks

(a) Let $x \in G$. Show that $Z_G(x) = \{g \in G \mid gx = xg\}$ is a subgroup of G .

5 marks

(b) Let A, B be abelian subgroups of G . Show that the subgroup $A \cap B$ is contained in the center of the subgroup $\langle A \cup B \rangle$.

6. Cosets and Quotients.

5 marks

- (a) Let G be a group, $A, B < G$. Show that the map $F: G/(A \cap B) \rightarrow (G/A) \times (G/B)$ given by $F(g(A \cap B)) = (gA, gB)$ is well-defined and one-to-one. Conclude that if A, B have finite index in G then so does $A \cap B$.

(problem continues on the next page)

5 marks

- (b) Suppose, in addition, that A, B are normal in G and that the indices $[G : A], [G : B]$ are finite and relatively prime. Show that F is an isomorphism of groups.

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