Midterm Exam Duration: 80 minutes

*This test has **6 questions** on **10 pages**, for a total of 60 points.*

- Do not turn this page over. You will have 80 minutes for the exam (between 14:00-15:20)
- This is a closed-book examination: no books, notes or electronic devices of any kind.
- You must justify all answers, regardless of the "operative word". Write in complete English sentences; proofs must be clear and concise.
- If you are using a result from the textbook, the lectures or the problem sets, state it properly.
- This exam is printed double-sided with the last two pages blank.

First Name: ________________ Last Name: ________________

Student-No: ________________ Signature: ________________

Question:	1	2	3	4	5	6	Total
Points:	15	10	5	10	10	10	60
Score:							

1. **Definition**

5 marks (a) Let G be a group. Define a *subgroup* H of G.

6 marks (b) Let G be a group and let $\{H_i\}_{i=1}^k$ be subgroups of G. Show that $\bigcap_{i=1}^k H_i$ is a subgroup of G.

4 marks (c) Let G, H be groups and let $f \in \mathrm{Hom}(G, H)$. Show that the image $\mathrm{Im}(f)$ is a subgroup of H.

2. **Calculation**

$\boxed{\text{5 marks}}$ (a) Solve the system of equations in $\mathbb{Z}/8\mathbb{Z}$:

$$\begin{cases} [5]x + [3]y &= [2] \\ [7]x - [2]y &= [3] \end{cases}$$

$\boxed{\text{5 marks}}$ (b) Find the cycle decomposition of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 7 & 3 & 1 & 8 & 9 & 6 & 4 \end{pmatrix}$.

No justification is needed for this problem.

3. **Groups**. In each case, decide if the assertion is true. If so, prove the assertion. If not, show where why definition fails to hold.

$\boxed{\text{2 marks}}$ (a) Let $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbb{R} \right\}$ and let $\cdot$ be matrix multiplication. Then $(N, \cdot)$ is a group.

$\boxed{\text{3 marks}}$ (b) Let $X = \mathbb{R} \setminus \{0\}$ be the set of non-zero reals, and let $f \colon X \to X$ be the map $f(x) = 2x$. Let $\cdot$ be the usual multiplication of real numbers. Then there is an operation $\star$ on X such that $f \colon (X, \cdot) \to (X, \star)$ is a group isomorphism.

4. **Groups and Permutations**

5 marks (a) Let G be a finite group of order n, and let $g \in G$. Without using Lagrange's Theorem, show that g has finite order.

5 marks (b) By Lagrange's Theorem, the order d of g divides n. Define a permutation of the set G by $\sigma(x) = gx$ (you don't need to show that this is a permutation). Find the cycle structure of this permutation, that is number of cycles of each length.

5. **Commutativity**.

Fix a group G.

$\boxed{\text{5 marks}}$ (a) Let $x \in G$. Show that $Z_G(x) = \{g \in G \mid gx = xg\}$ is a subgroup of G.

$\boxed{\text{5 marks}}$ (b) Let A, B be abelian subgroups of G. Show that the subgroup $A \cap B$ is contained in the center of the subgroup $\langle A \cup B \rangle$.

6. **Cosets and Quotients**.

5 marks (a) Let G be a group, $A, B < G$. Show that the map $F\colon G/(A \cap B) \to (G/A) \times (G/B)$ given by $F(g(A \cap B)) = (gA, gB)$ is well-defined and one-to-one. Conclude that if A, B have finite index in G then so does $A \cap B$.

(problem continues on the next page)

5 marks

(b) Suppose, in addition, that A, B are normal in G and that the indices $[G : A]$, $[G : B]$ are finite and relatively prime. Show that F is an isomorphism of groups.

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