

Math 322: Problem Set 9 (due 19/11/2015)

- P1. In class we classified the groups of order 12, finding the isomorphism types A_{12} , C_{12} , $C_4 \times C_3$, $C_2 \times C_6$, $C_2 \times S_6$. The dihedral group D_{12} is a group of order 12 – where does it fall in this classification?
- P2. (Numerology) Let G be a group of order p^2q where p, q are prime.
- Show that, unless $q \equiv 1 \pmod{p}$, G has a unique p -Sylow subgroup and isn't simple.
 - Show that, unless $p^2 \equiv 1 \pmod{q}$, G has a unique q -Sylow subgroup and isn't simple.
 - Show that if $q \equiv 1 \pmod{p}$ and $p^2 \equiv 1 \pmod{q}$ then $p = 2, q = 3$ and G isn't simple.

Sylow's Theorems

Write P_p for a p -Sylow subgroup of G .

- Let G be a simple group of order $36 = 2^2 \cdot 3^2$.
RMK The idea of P2 shows that a group of order p^2q^2 isn't simple unless $p^2q^2 = 36$.
 - Show that G acts non-trivially on a set of size 4.
 - Use the kernel of the action to show G isn't simple after all.
- Let G be a group of order $255 = 3 \cdot 5 \cdot 17$.
 - Show that $n_{17}(G) = 1$.
 - (*b) Show that P_{17} is central in G .
 - (*c) Show that $n_5(G) = 1$.
 - (d) Show that P_5 is also central in G .
 - (e) Show that $G \simeq C_3 \times C_5 \times C_{17} \simeq C_{255}$.
- Let G be a group of order 140
 - Show that $G \simeq H \times C_{35}$ where H is a group of order 4.
 - (*b) Classify actions of C_4 on C_{35} and determine the isomorphism classes of groups of order 140 with $P_2 \simeq C_4$.
 - (**c) Classify actions of V on C_{35} and determine the isomorphism classes of groups of order 140 with $P_2 \simeq V$.
- Let G be a finite group, $P < G$ a Sylow subgroup. Show that $N_G(N_G(P)) = N_G(P)$ (hint: let $g \in N_G(N_G(P))$ and consider the subgroup gPg^{-1}).
- Let G be a finite group of order n , and for each $p|n$ let P_p be a p -Sylow subgroup of G .
 - Show that $G = \langle \bigcup_{p|n} P_p \rangle$.
 - Suppose that G_p has a unique p -Sylow subgroup for each p . Show that $G = \prod_p P_p$ (internal direct product).

(hint for 2b: conjugation gives a homomorphism $G \rightarrow \text{Aut}(P_{17})$).

(hint for 2c: let G act by conjugation on $\text{Syl}_5(G)$ and use part b).