3/12/2015

MT: half scored in [27, 40] /60

Last time: Solvable gps:

G solvable $\iff$ normal series $\{e\} \trianglelefteq G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ with G_{i+1}/G_i abelian.

Saw: ① G solvable $\Rightarrow$ $H < G$, G/N solvable.

② $N \triangleleft G$, if $N, G/N$ solvable so is G.

Examples: $B_n = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} < GL_n(F)$ (upper-triangular matrices)

HW: $U_n \triangleleft B_n$, $B_n/U_n \cong (F^\times)^n$

Solvability top-down: relabel series $G = G_0 \triangleright G_1 \triangleright G_2 \cdots \triangleright G_n = \{e\}$

want G_0/G_1 to be abelian, want for any $x, y \in G$ that $[\bar{x}, \bar{y}] = e$

$\bar{x} = xG_1$, $\bar{y} = yG_1$ images of x, y in G/G_1.

$\iff$ want $[x, y] \in G_1$.

$\Rightarrow$ G/G_1 abelian iff $G_1 \supset \{ [x,y] \mid x, y \in G \}$

iff $G_1 \supset \langle \{[x,y]\} \rangle \overset{\text{def}}{=} [G, G] = G'$

"derived subgp"

HW: If $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = \{e\}$

G_i/G_{i+1} abelian then $G_i \supseteq G^{(i)}$: $G^{(0)} = G$, $G^{(i+1)} = (G^{(i)})'$

clearly $G^{(i)}/G^{(i+1)}$ commutative: killed all commutators

Conclusion: G solvable iff $G^{(n)} = \{e\}$ for some n

$G^{(1)}$ generated by $[x,y]$

$G^{(2)}$ " " $[[x,y],[z,w]]$

checks $(B_n)' = U_n$

Corollary: $(GL_n(F))' \supset (B_n)' = U_n$

also: $\det(xyx^{-1}y^{-1}) = 1$, so $[x,y] \in \mathrm{Ker}(\det) = SL_n(F) = \{g \in GL_n(F) \mid \det g = 1\}$

G' normal: let $\varphi \in \mathrm{Aut}(G)$. Then $\varphi([x,y]) = \varphi(xyx^{-1}y^{-1})$

$$= \varphi(x)\varphi(y)\varphi(x^{-1})\varphi(y^{-1})$$
$$= [\varphi(x), \varphi(y)]$$

then φ permutes the commutators,

so fixes the subgp they generate: $\varphi(G') = G'$

What is the normal subgp generated by $U_n(F)$?

over $\mathbb{R}$, $U_n(\mathbb{R})$ = upper triangular +1-diagonal, $\overline{U}_n$ = lower-triangular +1-diagonal

$$\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & & \\ * & 1 & \\ * & * & 1 \end{pmatrix}$$

jointly generate $SL_n(\mathbb{R})$ (Gaussian elimination)

Group theory in topology

Topology: study of properties unchanged by deformation

Basic question: given X, Y. want to know: are they the same?

distinguish X, Y say, by dimension

let ~~X~~ be fundamental group

$(X,*)$ is a (topological space) + pt $*$

consider the set of based loops $C((S^1,*), (X,*))$

(cts maps $\gamma: [0,1] \to X$, s.t. $\gamma(0) = *$, $\gamma(1) = *$)

natural operation: concatenation: if γ_1, γ_2 loops define

$\gamma_1 \cdot \gamma_2$: do γ_1, then γ_2.

associative, $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$

identity: constant loop $e(t) = *$ for all t.

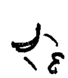
"Inverse": $\bar{\gamma}(t) = \gamma(1-t)$ (reverse direction)

declar $\gamma_1 \sim \gamma_2$ if can deform γ_1 to γ_2 (endpoints fixed)

deform $\gamma \cdot \bar{\gamma}$ to e: at time 0

at time ε

at time $1-\varepsilon$



at time 1

<u>Combinatorics</u>

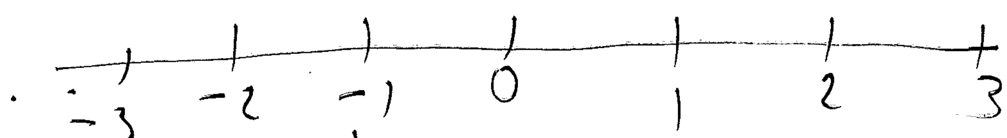
G gp, S generating set. (assume symmetric: if $s \in S$, $s^{-1} \in S$ too)

Def: the Cayley graph $Cay(G;S)$ is the graph with vertex set G

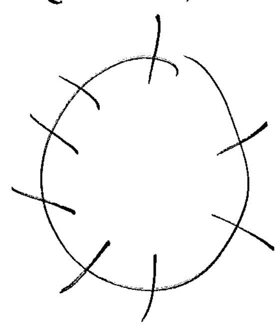
edges $E = \{(g, gs) \mid g \in G, s \in S\}$

(Graph: pair $\Gamma = (V, E)$ V vertices, E: edges connect vertices

Examples $Cay(\mathbb{Z}, \{\pm 1\})$ edges: $(n, n \pm 1)$



$Cay(\mathbb{Z}/n\mathbb{Z}, \{\pm 1\}) =$



n-cycle

if $(g, gs) \in E$ then $(gs, (gs)s^{-1}) = (gs, g) \in E$ too

G acts on $Cay(G;S)$: translation $g \cdot x = gx$

maps edge (x, xs) to edge $(gx, (gx)s)$

Example: (subject to a congruence condition) $SL_2(\mathbb{F}_q)$, q prime,

has a generating set S of size $p+1$ with $Cay(SL_2(\mathbb{F}_q); S)$

extremely well-connected:

This network has $\sim q^3$ vertices, each with only $p+1$ neighbours

(think p fixed, $q \to \infty$) but if cut it into two pieces A, B

many connections across the cut.

check ① $\sim$ is an equiv rel

② if $\gamma_1 \sim \gamma_1'$, $\gamma_2 \sim \gamma_2'$ then $\gamma_1 \cdot \gamma_2 \sim \gamma_1' \cdot \gamma_2'$

Follows: • still well-defined on $C((S^1,*),(X,*))/\sim$

$[e]$ still identity

now $[\gamma]\cdot[\bar{\gamma}] = [\gamma\cdot\bar{\gamma}] = [e]$

so we got a group! Call it $\pi_1(X,*)$.

Examples: $\pi_1(S^1) = \mathbb{Z}$ (γ loop on S^1, need to know only winding number)

$\pi_1(S^2) = \{e\}$

$\pi_1(S^1 \times S^1) = \mathbb{Z}\times\mathbb{Z} \cong \mathbb{Z}^2$

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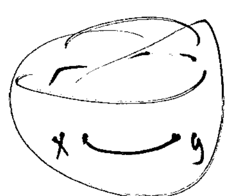
suppose x, y points in X connected by path p



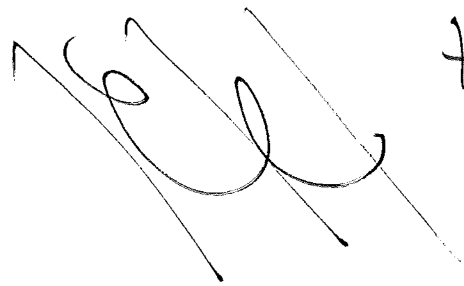
given loop γ based at y the loop $p\cdot\gamma\cdot\bar{p}$ is based at x

converse identification uses $\bar{p}$, these are inverse modulo deformation:

$\pi_1(X,x) \cong \pi_1(X,y)$

$\pi_1(X,x) = \left(C((S^1,*),(X,*))/\sim, \ \cdot_\sim \right)$

$\pi_1(X\times Y) \cong \pi_1(X)\times\pi_1(Y)$



$\mathbb{H} = \{ z = x+iy \mid y>0 \}$

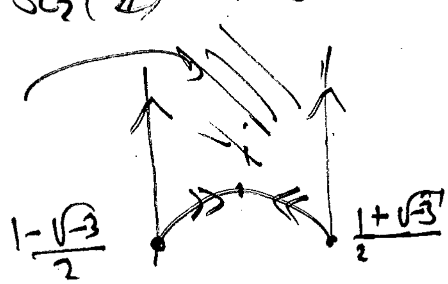
$g \in SL_2(\mathbb{R})$: $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ define $g\cdot z = \frac{az+b}{cz+d}$

ex: this is a gp action on $\mathbb{H}$

(preserve distance) action transitive, $\mathrm{Stab}_{SL_2(\mathbb{R})}(i) = SO(2)$

In particular, $SL_2(\mathbb{Z})$ acts

$SL_2(\mathbb{Z})\backslash\mathbb{H}$



$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} z = z+1$

$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} z = -\frac{1}{z}$

examples count solutions to $x^2+y^2+z^2+w^2=N$ $((x,y,z,w)\in\mathbb{Z}^4)$
using holomorphic fcns on $SL_2(\mathbb{Z})\backslash\mathbb{H}$