MATH 101: INTEGRATION USING PARTIAL FRACTIONS

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In this note I collect a few examples of computing indefinite integrals by expansion into partial fractions.

Summary of the method for finding the expansion.

(1) If the degree of the numerator is not smaller than that of the denominator – perform *long division* [review CLP notes for how to do this]
(2) *Factor* the denominator.
(3) Repeatedly do the following:
   (a) For each "bad point" a (zero of the denominator, that is a factor of the denominator of form $(x-a)^k$), plug in a into the numerator and all other factors of the denominator, to obtain an asymptotic of the form
   $$\lim_{x\to a}(x-a)^k f(x) = A\,.$$
   where A is a numerical constant.
   (b) Subtract each such "partial fraction" $\frac{A}{(x-a)^k}$ from $f(x)$, bring to a common denominator and *cancel* factors of $(x-a)$ for each a.
      - Now the power of $(x-a)$ in the denominator has gone down.
   (c) Return to part (a) until all partial fractions are found.
(4) After subtraction, the only remaining factors of the denominator will be irreducible quadratics and their powers. *Promise in Math 101*: there will be at most one such factor.

Summary of integration formulas for the partial fractions.

(1) $\int \frac{A}{x-a}\,\mathrm{d}x = A\log|x-a| + C$
(2) $\int \frac{A}{(x-a)^k}\,\mathrm{d}x = -\frac{A}{k-1}\frac{1}{(x-a)^{k-1}}$ $(k \geq 2)$
(3) $\int \frac{Ax+B}{ax^2+bx+c}$: write the numerator in the form $\frac{A}{2a}(2ax+b) + \left(B - \frac{Ab}{2a}\right)$, and complete the square in the denominator to get $ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}$. Conclude that
$$\int \frac{Ax+B}{ax^2+bx+c} = \frac{A}{2a}\int \frac{(2ax+b)\,\mathrm{d}x}{ax^2+bx+c} + \left(\frac{2aB-Ab}{2a^2}\right)\int \frac{\mathrm{d}x}{\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}}\,.$$
The first integral is immediate (u-substitution) and for the second use an inverse trig substitution ($x + \frac{b}{2a} = \frac{\sqrt{4ac-b^2}}{2a}\tan\theta$).

Summary of the examples below. Problems 1-3 are taken from past finals. Problems 4-5 are intended as all-steps example

Problem 1 (Final, 2010)**.** Evaluate $\int \frac{x^2-9}{x(x^2+9)}\,\mathrm{d}x$.

Solution: Step 0: the degree of the numerator is less than the degree of the numerator.

(1) The denominator is already factored.

(2) At zero we have $\lim_{x\to 0} \frac{x^2-9}{x^2+9} = \frac{-9}{9} = -1$, so we expect a term $-\frac{1}{x}$ in the expansion. We subtract that from the original function to get:

$$\frac{x^2-9}{x(x^2+9)} + \frac{1}{x} = \frac{x^2-9+(x^2+9)}{x(x^2+9)} = \frac{2x^2}{x(x^2+9)} = \frac{2x}{x^2+9}$$

so that

$$\frac{x^2-9}{x(x^2+9)} = -\frac{1}{x} + \frac{2x}{x^2+9}\,.$$

(3) We finally compute the integral

$$\begin{aligned}\int \frac{x^2-9}{x(x^2+9)}\,\mathrm{d}x &= -\int \frac{1}{x}\,\mathrm{d}x + \int \frac{2x}{x^2+9}\,\mathrm{d}x \\ &= -\log|x| + \int \frac{d\left(x^2+9\right)}{x^2+9} \\ &= -\log|x| + \log\left(x^2+9\right) + C\,.\end{aligned}$$

Problem 2 (Final, 2007)**.** Evaluate $\int_0^1 \frac{2x+3}{(x+1)^2}\,\mathrm{d}x$.

Solution: The degree of the numerator is less than the degree of the denominator and the denominator is factored. Near $x=-1$ we have

$$\lim_{x\to -1} \frac{2x+3}{1} = \frac{2(-1)+3}{1} = 1$$

and

$$\frac{2x+3}{(x+1)^2} - \frac{1}{(x+1)^2} = \frac{2x+2}{(x+1)^2} = \frac{2(x+1)}{(x+1)^2} = \frac{2}{(x+1)}$$

so that

$$\frac{2x+3}{(x+1)^2} = \frac{1}{(x+1)^2} + \frac{2}{x+1}$$

and

$$\begin{aligned}\int_0^1 \frac{2x+3}{(x+1)^2}\,\mathrm{d}x &= \left[-\frac{1}{x+1} + 2\log|x+1|\right]_{x=0}^{x=1} \\ &= \left[-\frac{1}{2} + 2\log 2\right] - \left[-1 + 2\log 1\right] \\ &= \frac{1}{2} + 2\log 2\,.\end{aligned}$$

Problem 3 (Final, 2007)**.** Write the form of the partial-fraction decomposition for $\frac{10}{(x+1)^2(x^2+9)}$. Do not determine the numerical values of the coefficients.

Solution: $\frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{Cx+D}{x^2+9}$

Remark. We note that x^2+9 is irreducible, and that because it's quadratic the numerator can be linear and not just a constant.

Problem 4. Find the partial fractions expansion of $\frac{2x^3+7x^2+6x+1}{x^3+x^2+x}$

$$\frac{2x^3+7x^2+6x+1}{x^3+x^2+x}$$

Solution: Step 0: the numerator is of the same degree as the denominator, so we divide: $2x^3+7x^2+6x+1-2\left(x^3+x^2+x\right)=5x^2+4x+1$ so

$$2x^3+7x^2+6x+1=2\left(x^3+x^2+x\right)+\left(5x^2+4x+1\right)$$

and

$$\frac{2x^3+7x^2+6x+1}{x^3+x^2+x}=2+\frac{5x^2+4x+1}{x^3+x^2+x}$$

Step 1: we factor the denominator; $x^3+x^2+x=x(x^2+x+1)$ and x^2+x+1 is irreducible since it has discriminant $1-4=-3$.

Step 2: Near zero we have

$$\lim_{x\to 0}\frac{5x^2+4x+1}{x^2+x+1}=1\,.$$

Subtracting we find

$$\frac{5x^2+4x+1}{x(x^2+x+1)}-\frac{1}{x}=\frac{\left(5x^2+4x+1\right)-\left(x^2+x+1\right)}{x(x^2+x+1)}=\frac{4x^2+3x}{x(x^2+x+1)}=\frac{4x+3}{x^2+x+1}$$

so we finally have

$$\frac{2x^3+7x^2+6x+1}{x^3+x^2+x}=2+\frac{1}{x}+\frac{4x+3}{x^2+x+1}\,.$$

Problem 5. Find the partial fraction expansion of $\frac{x^5+2}{x^2(x+1)^2}$.

Solution: Step 0: The degree of the numerator is greater than that of the denominator, so we need to divide. The denominator is $x^4+2x^3+x^2$, so we have:

$$\begin{aligned} x^5+2 &= x\left(x^4+2x^3+x^2\right)-\left(2x^4+x^3\right)+2 \\ &= x\left(x^4+2x^3+x^2\right)-2\left(x^4+2x^3+x^2\right)+3x^3+2x^2+2 \end{aligned}$$

so

$$\frac{x^5+2}{x^2(x+1)^2}=\frac{3x^3+2x^2+2}{x^2(x+1)^2}+(x-2)\,.$$

(1) The denominator is already factored.
(2) We have

$$\begin{aligned} \lim_{x\to 0}\frac{3x^2+2x^2+2}{(x+1)^2} &= \frac{2}{1}=2 \\ \lim_{x\to -1}\frac{3x^2+2x^2+2}{x^2} &= \frac{-3+2+2}{1}=1 \end{aligned}$$

so get the terms

$$\frac{2}{x^2}+\frac{1}{(x+1)^2}\,.$$

(3) Subtracting, we have

$$\begin{aligned} \frac{3x^3+2x^2+2}{x^2(x+1)^2}-\left(\frac{2}{x^2}+\frac{1}{(x+1)^2}\right) &= \frac{\left(3x^3+2x^2+2\right)-2\left(x+1\right)^2-x^2}{x^2(x+1)^2} \\ &= \frac{3x^3+2x^2+2-2x^2-4x-2-x^2}{x^2(x+1)^2} \\ &= \frac{3x^3-x^2-4x}{x^2(x+1)^2}=\frac{3x^2-x-4}{x(x+1)^2} \\ &= \frac{(x+1)(3x-4)}{x(x+1)^2}=\frac{3x-4}{x(x+1)}\,. \end{aligned}$$

(4) We now repeat the process:

$$\begin{aligned}\lim_{x\to 0} \frac{3x-4}{(x+1)} &= -4 \\ \lim_{x\to -10} \frac{3x-4}{x} &= 7\end{aligned}$$

so we get

$$\frac{3x-4}{x(x+1)} = -\frac{4}{x} + \frac{7}{x+1}\,.$$

(5) It follows that

$$\frac{x^5+2}{x^2(x+1)^2} = \frac{2}{x^2} - \frac{4}{x} + \frac{1}{(x+1)^2} + \frac{7}{x+1}\,.$$