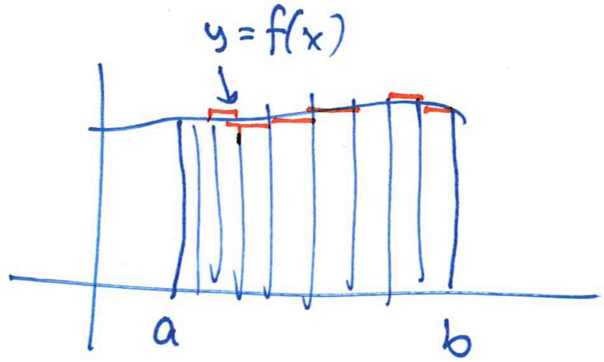
3. The definite integral (9/1/2017)

Goals.

(1) Define the definite integral
(2) Convert between integrals and Riemann sums
(3) Properties of Σ
(4) Evaluate an integral by definition
(5) Evaluate integrals by realizing them as areas
(6) Properties of the integral

Last Time.

Area under curve via vertical slices:



$\Delta x = \frac{b-a}{n}$

$x_i = a + i\Delta x$

chose $x_{i-1} \le x_i^* \le x_i$

approximated

$$\text{Area} \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

area of each strip

sum over strips

<u>fact:</u> $\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$

(if f is cts, the limit always exists)

<u>Def:</u> The <u>integral</u> of f on $[a,b]$ is this limit:

$$\int_a^b f(x)\,dx \stackrel{def}{=} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Math 101 – WORKSHEET 3
THE DEFINITE INTEGRAL

(1) (Sums) Given $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ find

(a) $\sum_{i=1}^{2n} i$

(b) $\sum_{i=1}^{n} (2i)$

(2) (Riemann sums)

(a) Express the area between the x-axis, the lines $x = 1$ and $x = 4$ and the graph of $f(x) = \cos(x^2)$ as a limit. Use the right-hand rule.

$a=1, b=4$
$\Delta x = 3/n$, Area $= \lim_{n\to\infty} \sum_{i=1}^{n} \cos\left(\left(1+\frac{3i}{n}\right)^2\right)\frac{3}{n}$

$x_i = 1 + \frac{3i}{n}$

right-hand rule

(b) Express $\lim_{n\to\infty} \frac{1}{2n} \sum_{i=1}^{n} \tan\left(\frac{i}{3n}\right)$ as an integral and as an area.

$f(x_i) = \frac{3}{2}\tan(x_i)$

$x_i = \frac{i}{3n} = a + i\Delta x \Rightarrow a = 0$, $\Delta x = 1/3n$

$\tan(x_i)\cdot\frac{1}{2n} = f(x_i)\Delta x$

limit $= \int_0^{1/3} \left(\frac{3}{2}\tan x\right) dx$

$b = 1/3$

$b - a = 3n\,\Delta x$

alternative

$\Delta x = \frac{1}{2n}$

$\frac{i}{3n} = \frac{2}{3}\cdot\frac{i}{2n} = \frac{2}{3} i\Delta x$

$f(x) = \tan\left(\frac{2}{3}x\right)$

$\int_0^{1/2} \tan(2/3 x)\, dx$

Remark. For any choice of Δx (proportional to $\frac{1}{n}$) and any choice of a, there is a solution, and they are all correct. The first choice is perhaps the most natural one, but there is no one single answer to this problem. Those who already know about "change of variables" in integrals can see how all the answers are related.

Date: 9/1/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

One example

Say we want to compute $\int_1^4 x^2 dx$

Here $\Delta x = \frac{3}{n}$, $x_i = 1 + \frac{3i}{n}$, $f(x_i) = \left(1 + \frac{3i}{n}\right)^2 = 1 + \frac{6i}{n} + \frac{9i^2}{n^2}$

right-side rule

Riemann sum:

$$\int_1^4 x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(1 + \frac{6i}{n} + \frac{9i^2}{n^2}\right)}_{f(x_i)} \cdot \underbrace{\frac{3}{n}}_{\Delta x} \quad \leftarrow \text{definition}$$

$$= \lim_{n\to\infty} \left(\frac{3}{n} \sum_{i=1}^{n} 1 + \frac{3}{n} \cdot \frac{6}{n} \sum_{i=1}^{n} i + \frac{3}{n} \cdot \frac{9}{n^2} \sum_{i=1}^{n} i^2 \right) \quad \leftarrow \text{algebra}$$

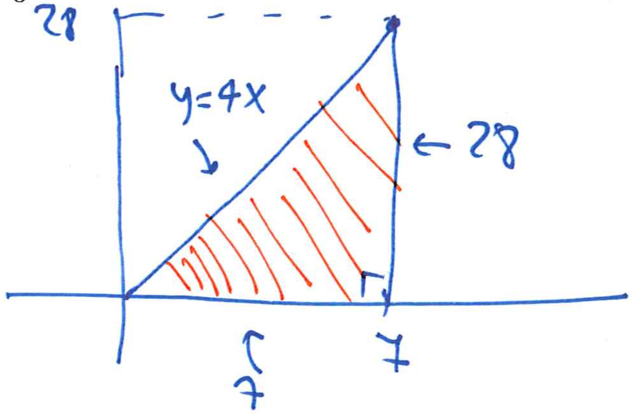
$$= \lim_{n\to\infty} \left(\frac{3}{n} \cdot n + \frac{18}{n^2} \frac{n(n+1)}{2} + \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6} \right) \quad \leftarrow \text{summation formula}$$

$$= \lim_{n\to\infty} \left(3 + 9 \cdot \frac{n^2+n}{n^2} + \frac{9}{2} \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \right) \quad \leftarrow \text{took limit}$$

$$= 3 + 9 + 9 = 21.$$

(3) Evaluate

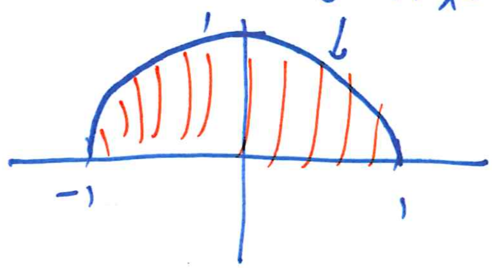
(a) $\int_0^7 4x\,dx$



Area $= \frac{1}{2} \cdot 7 \cdot 28 = 98.$

(b) $\int_{-1}^{1} \sqrt{1-x^2}\,dx$

$y = \sqrt{1-x^2} \iff x^2+y^2=1$



semicircle surrounds area of $\frac{1}{2}\pi$

(c) $\int_{-2}^{2}(3+x)\,dx$ $= \int_{-2}^{2} 3\,dx + \int_{-2}^{2} x\,dx = 12 + 0 = 12$

function is odd
interval symmetric