20. Centre of Mass (27/2/2017)

Goals:

(1) Average value: not covered in class; worksheet posted to section website
(2) Centre of mass
  (a) Discrete particle system: averaging
  (b) Formulas to memorize

Center of Mass: Distribution of masses
Each piece "votes" for its location, average is weighted.

Worksheet: discrete pts

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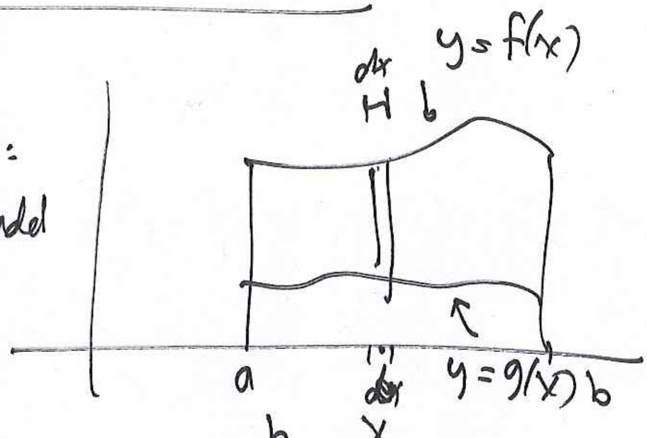
cts distribution: chop up and add



slice at x votes for CM to be at x.
weight: $(f(x)-g(x))\cdot dx$

x-co-ord of CM: $\frac{1}{\text{Area}}\int_a^b (f(x)-g(x))\cdot x\,dx$

$\text{Area} = \int_a^b (f-g)\,dx$

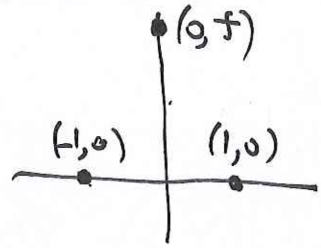
y-co-ord of CM: $\frac{1}{\text{Area}}\int_a^b (f(x)-g(x))\cdot\frac{f(x)+g(x)}{2}\,dx = \frac{1}{2\cdot\text{Area}}\int_a^b (f^2(x)-g^2(x))\,dx$

Math 101 – WORKSHEET 20
THE CENTRE OF MASS

1. Point masses

(1) Three masses are placed at the points $(-1,0)\,,(1,0)\,,(0,5)$. Find the centre of mass of the configuration.

(a) When the masses are equal,



$$\tfrac{1}{3}(-1,0) + \tfrac{1}{3}(1,0) + \tfrac{1}{3}(0,5) = (0, 5/3)$$

symmetry

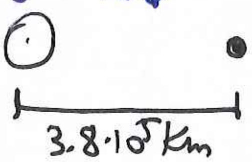
(b) When the mass at $(-1,0)$ is twice as large as the others.

$$\tfrac{1}{2}(-1,0) + \tfrac{1}{4}(1,0) + \tfrac{1}{4}(0,5) = \left(-\tfrac{1}{4}, \tfrac{5}{4}\right)$$

$\frac{\text{weight of pt}}{\text{total weight}}$

(2) The mass of the Earth is about 6×10^{24}kg. The mass of the Moon is about 7.2×10^{22}kg. The distance between the centres of the Earth and the Moon is $3.8 \cdot 10^5$km. Where is the centre of mass of the Earth–Moon system? [aside: the radius of the Earth is about 6400km].

put origin here



distance from centre of earth: $\frac{6\cdot 10^{24}}{6.072\cdot 10^{24}}\cdot 0 + \frac{7.2\cdot 10^{22}}{6.072\cdot 10^{24}}\cdot 3.8\cdot 10^{5}\,\text{Km} \approx 1.2\cdot 3.8\cdot 10^{3}\,\text{Km} \approx 4{,}500\,\text{Km}$

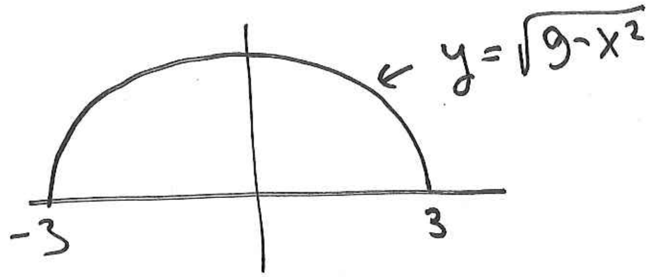
(3) A tenderizing hammer consists of a 1kg head attached to a 30cm-long shaft massing 400g.

(a) Find the centre of mass of the hammer.

(b) What fraction of the mass of the hammer is on each side of the centre of mass?

Date: 27/2/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Example: Semicircle of radius 3



Area: $\frac{1}{2} \cdot \pi \cdot 3^2 = \frac{9}{2}\pi$

x-co-ord of com at 0 by symmetry, or:

$$\frac{1}{\text{Area}} \cdot \int_{-3}^{3} x \cdot \sqrt{9-x^2}\, dx = 0$$

odd integrand

y-co-ord:

$$\frac{1}{2 \cdot \text{Area}} \cdot \int_{-3}^{3} \left(\sqrt{9-x^2}^{\,2}\right) dx = \frac{1}{9\pi} \int_{-3}^{3} (9-x^2)\, dx =$$

symmetry

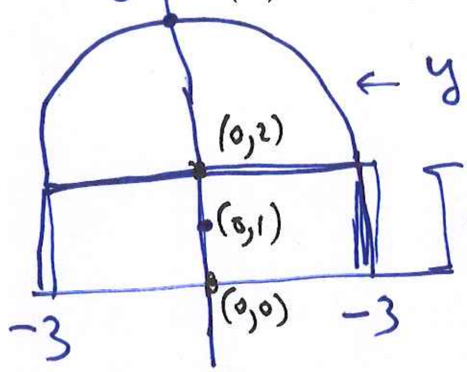
$$= \frac{2}{9\pi} \left[9x - \frac{x^3}{3} \right]_{x=0}^{x=3} = \frac{4}{\pi}.$$

symmetry

2. Regions

(3) (Final 2013) The region R consists of a semicircle of radius 3 on top of a rectangle of width 6 and height 2. Find its centre of mass.

(a) Using the formulas above



$\leftarrow y = \sqrt{9-x^2} + 2$ $\quad X_{CM} = 0$ by symmetry.

Area $= \frac{9}{2}\pi + 12$

$$Y_{CM} = \frac{1}{2\left(\frac{9}{2}\pi + 12\right)} \cdot \int_{-3}^{3} \left(\sqrt{9-x^2}+2\right)^2 dx =$$

$$= \frac{1}{9\pi+24} \int_{-3}^{3} \left(9 - x^2 + 4\sqrt{9-x^2} + 4\right) dx = \frac{1}{9\pi+24} \cdot \int_{-3}^{3} \left(13 - x^2 + 4\sqrt{9-x^2}\right) dx$$

$$= \frac{1}{9\pi+24}\left(13\cdot 6 - \frac{2\cdot 3^3}{3} + 4\cdot\frac{9}{2}\pi\right) = \frac{18\pi+60}{9\pi+24}$$

(b) Using the known locations of the centres of mass of the semicircle and the rectangle.

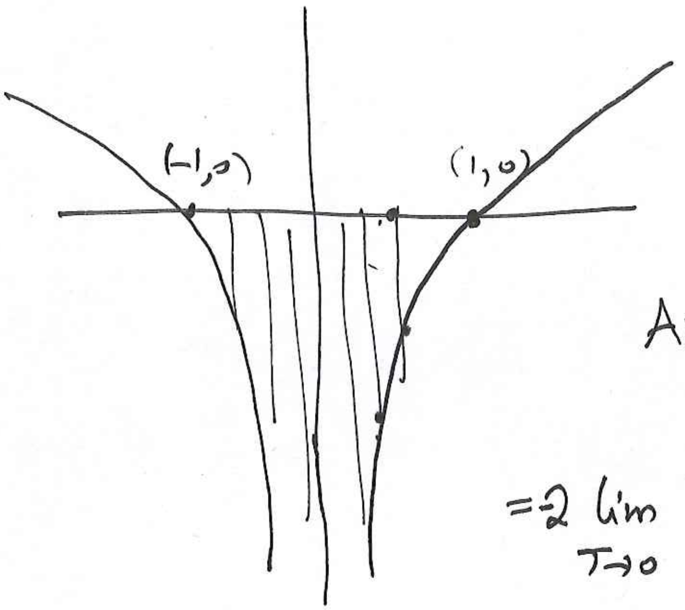
rectangle / total area

semicircle / total area

$$\frac{12}{\frac{9}{2}\pi + 12} \cdot 1 + \frac{\frac{9}{2}\pi}{\frac{9}{2}\pi + 12} \cdot \left(2 + \frac{4}{\pi}\right) = \ldots = \frac{18\pi + 60}{9\pi + 24}$$

CM of rectangle

(4) Find the centre of mass of the region lying below the x axis, between the branches of $\log |x|$.



$$\text{Area} = 2 \cdot \int_0^1 (-\log x)\, dx$$

$$= -2 \lim_{T \to 0} \int_T^1 \log x \, dx = -2 \lim_{T \to 0} \left[x \log x - x \right]_T^1$$

by parts

$$= -2 \lim_{T \to 0} \left[-1 - T\log T + T \right] = 2 - \lim_{T \to 0} T \log T$$

$$= 2$$

l'Hôpital

$$T \log T = \frac{\log T}{1/T}$$