36. Review Session II (5/3/2017)

On the final

- Friday, April 21 between 12:~~30~~00-~~15:00~~ 14:30 at SRC A
  - Assigned seating
  - Two versions
  - Strict exam conditions (no baseball caps, pencil cases, digital watches)
- 150 minutes, 75 points ($\sim$ 2minutes /point)
- Structure
  - 2 pages of "answer in box" (8 questions)
  - ~~3~~ 4 pages of short-answer questions (~~6~~ 8 questions)
  - 5 long-answer questions
- Cumulative (slight emphasis on material after Quiz 5)

Preparing for final

- Practice exams from this year & last } have same structure
- Last" year's final

Use (at least some) by writing under exam conditions

- Many past finals on dept website
- Solution for some on MER wiki.
- Office hours (T: 10-11, Wed: 12-12:30), Piazza, MLC

Question: Find the MacLaurin series of $\int_0^x \sin(5t^2)\,dt$?

Instead, $\int_0^x \arctan(5t^2)\,dt$?

Solution: Recall that $\arctan(u) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} u^{2n+1}$ ← memorize std expansion

Thus $\arctan(5t^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} (5t^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5^{2n+1}}{2n+1} \cdot t^{4n+2}$ ← plug in value

We may integrate power series term-by-term,

so

$$\int_0^x \arctan(5t^2)\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}}{2n+1} \cdot \int_0^x t^{4n+2}\,dt$$

$$\int_0^x t^a\,dt = \left[\frac{t^{a+1}}{a+1}\right] = \frac{1}{a+1} x^{a+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5^{2n+1}}{2n+1} \cdot \frac{1}{4n+3} x^{4n+3}$$

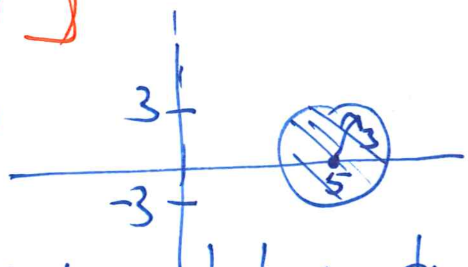
i.e.

$$\boxed{\int_0^x \arctan(5t^2)\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5^{2n+1}}{(2n+1)(4n+3)} x^{4n+3}}$$

Problem: Find the volume of the solid obtained by rotating the disc $\{(x-5)^2+y^2 \le 9\}$ about the y-axis.

[difficulty in realizing this is the disc bounded by the circle $(x-5)^2+y^2=9$]

Solution: Let's draw a picture



Express the disc as the region between two curves:

For each y, the points x range between left & right semicircles:
to find x, on boundary we have $y^2+(x-5)^2=9$ so

$$(x-5)^2 = 9-y^2 \quad \text{so} \quad x-5=\pm\sqrt{9-y^2}$$

so x range between $5-\sqrt{9-y^2}$ and $5+\sqrt{9-y^2}$, so the volume is

$$a^2-b^2=(a-b)(a+b)$$

$$\pi\int_{y=-3}^{y=3}\Big[\underbrace{\big(5+\sqrt{9-y^2}\big)^2}_{\text{outer radius}}-\underbrace{\big(5-\sqrt{9-y^2}\big)^2}_{\text{inner radius}}\Big]dy \stackrel{\downarrow}{=} \pi\int_{-3}^{3}\Big[2\sqrt{9-y^2}\Big]\Big[10\Big]dy$$

$= 20\pi\int_{-3}^{3}\sqrt{9-y^2}\,dy$ but $\int_{-3}^{3}\sqrt{9-y^2}\,dy$ is the area under a semicircle of radius 3, that is $\frac{1}{2}\cdot\pi\cdot 3^2$.

The volume is then $20\pi\cdot\frac{1}{2}\cdot\pi\cdot 9 = 90\pi^2$

Alternative:

$$\left(5+\sqrt{9-y^2}\right)^2-\left(5-\sqrt{9-y^2}\right)^2=\left(25+9-y^2+2\cdot 5\cdot\sqrt{9-y^2}\right)$$
$$-\left(25+9-y^2-2\cdot 5\sqrt{9-y^2}\right)$$
$$=20\sqrt{9-y^2}$$

Problem: Evaluate $\displaystyle\int \frac{2x-1}{x^2-2x+5}\,dx$

Solution: $x^2-2x+5=(x^2-2x+1)+4=(x-1)^2+2^2$

so $$\int \frac{2x-1}{x^2-2x+5}\,dx=\int\frac{2(x-1)}{(x-1)^2+4}\,dx+\int\frac{dx}{(x-1)^2+4}$$

$\leftarrow$ $2x-1=2x-2+1$
$=2(x-1)+1$

let $u=(x-1)^2+4$
$du=2(x-1)$

$x-1=2v$
$v=\frac{x-1}{2}$
$dv=\frac{1}{2}dx$

$$=\int\frac{du}{u}+\int\frac{2\,dv}{4v^2+4}=\log|u|+\frac{1}{2}\int\frac{dv}{1+v^2}=\log|u|+\frac{1}{2}\arctan(v)+C$$
$$=\log\left((x-1)^2+4\right)+\frac{1}{2}\arctan\left(\frac{x-1}{2}\right)+C$$

More complicated:

$$\frac{x^4+1}{(x-2)(x^2-2x+5)} = \frac{x^4+1}{x^3-4x^2+9x-10}$$

start ~~B~~ with a division

$$\begin{array}{r|l}
 & x+4 \\
\hline
x^3-4x^2+9x-10 & x^4 \qquad +1 \\
 & -\;x^4-4x^3+9x^2-10x \\
\hline
 & 4x^3-9x^2+10x+1 \\
 & -\;4x^3-16x^2+36x-40 \\
\hline
 & 7x^2-26x+41
\end{array}$$

so $$\frac{x^4+1}{x^3-4x^2+9x-10} = x+4+\frac{7x^2-26x+41}{(x-2)(x^2-2x+5)}$$

the partial fraction expansion will have a term $\frac{A}{x-2}$

with $$A = \lim_{x\to 2}\frac{7x^2-26x+41}{x^2-2x+5} = \frac{28-52+41}{5} = \frac{17}{5}$$

so $$\frac{x^4+1}{(x-2)(x^2-2x+5)} = x+4+\frac{17/5}{x-2}+\frac{Bx+C}{x^2-2x+5}$$ and

$$\frac{Bx+C}{x^2-2x+5} = \frac{7x^2-26x+41}{(x-2)(x^2-2x+5)} - \frac{17/5}{x-2}$$