Math 101 – SOLUTIONS TO WORKSHEET 24
SERIES

1. Review: Geometric and telescoping series

(1) Decide whether the following series converge or diverge
  (a) $\sum_{n=5}^{\infty} \frac{\pi^{2n+3}}{9^{n-2}}$
  Solution: This is a geometric series with ratio $\frac{\pi^2}{9} = \left(\frac{\pi}{3}\right)^2 > 1$ so the terms escape to infinity and the series diverges.
  (b) $\sum_{n=5}^{\infty} \frac{e^{2n+2}}{9^{n-2}}$
  Solution: This is a geometric series with ratio $\frac{e^2}{9} = \left(\frac{e}{3}\right)^2 < 1$ so it is convergent.
  (c) $\sum_{n=1}^{\infty} \left(n^2 - (n+1)^2\right)$
  Solution: The nth partial sum is $\left(1^2 - 2^2\right) + \left(2^2 - 3^2\right) + \cdots + \left(n^2 - (n+1)^2\right) = 1^2 - (n+1)^2$ and these clearly tend to $-\infty$ as $n \to \infty$ so the series diverges.

2. Skill 1: elements of a convergent series

(2) Show the following series diverge
  (a) $\sum_{n=1}^{\infty} (-1)^n$
  Solution: The terms have magnitude 1, don't decay to zero.
  (b) $\sum_{n=0}^{\infty} n^2 \sin(n)$
  Solution: There are large values of n where $\sin(n)$ is close to 1 so $n^2 \sin(n)$ is large.
  (c) $\sum_{n=1}^{\infty} \frac{n+\sin n}{n}$
  Solution: $\lim_{n\to\infty} \frac{n+\sin n}{n} = 1 + \lim_{n\to\infty} \frac{\sin n}{n} = 1 \neq 0$.

3. Review of improper integrals

(3) Show the following series divergeShow that $\int_2^{\infty} \frac{\mathrm{d}x}{x}$ diverges.
  Solution: $\int_2^T \frac{\mathrm{d}x}{x} = \log T - \log 2 \xrightarrow[T\to\infty]{} \infty$

(3) Show that $\int_2^{\infty} \frac{\mathrm{d}x}{x^3+5}$ converges.
  Solution: For $x > 0$ we have $x^3 + 5 > x^3 > 0$ so $0 < \frac{1}{x^3+5} < \frac{1}{x^3}$ and $\int_2^T \frac{\mathrm{d}x}{x^3} = \frac{1}{2}\left(\frac{1}{2^2} - \frac{1}{T^2}\right) \xrightarrow[T\to\infty]{}$ $\frac{1}{8}$ so the integral converges by the comparison test.

(4) Evaluate $\int_0^{\infty} xe^{-x}\,\mathrm{d}x$.
  Solution: We integrate by parts:

$$\begin{aligned}\int_0^T xe^{-x}\,\mathrm{d}x &= \left[-xe^{-x}\right]_0^T - \int_0^T \left(-e^{-x}\right)\mathrm{d}x = -Te^{-T} + \left[e^{-x}\right]_0^T = 1 - Te^{-T} - e^{-T} \\ &= 1 - \frac{T+1}{e^T}\,.\end{aligned}$$

Now as $T \to \infty$ by l'Hôpital,

$$\lim_{T\to\infty} \frac{T+1}{e^T} = \lim_{T\to\infty} \frac{1}{e^T} = 0$$

so

$$\int_0^{\infty} xe^{-x}\,\mathrm{d}x = \lim_{T\to\infty} \int_0^T xe^{-x}\,\mathrm{d}x = 1\,.$$

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4. Skill 2: The Integral test

(6) Decide whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{n}$
Solution: $f(x) = \frac{1}{x}$ is **decreasing** and **positive** and $\int_2^{\infty} \frac{\mathrm{d}x}{x} = \infty$ so the series diverges.

(b) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (your answer will depend on p).
Solution: $f(x) = \frac{1}{x}$ is decreasing and $\int_2^{\infty} \frac{\mathrm{d}x}{x} = \infty$ so the series diverges.