Math 101 – SOLUTIONS TO WORKSHEET 25
THE INTEGRAL TEST

1. The integral test

(1) Decide if each series converges or diverges

(a) $\sum_{n=1}^{\infty} \frac{n}{e^n}$

Solution: Let $f(x) = xe^{-x}$, so that the series is $\sum_{n=1}^{\infty} f(n)$. Then $f(x) > 0$ for all x. Also, we have $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$ which is negative for $x > 1$ so f is **eventually decreasing**. We know that $\int_0^{\infty} xe^{-x}\,\mathrm{d}x$ converges (see previous worksheet) so by the integral test our series converges as well.

(b) (Final 2014) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ (your answer will depend on p!)

Solution: Suppose $p > 0$ (if $p \le 0$ compare with $\sum_{n=1}^{\infty} \frac{1}{n}$ – see next lecture) and let $f(x) = \frac{1}{x(\log x)^p}$ so that the series is $\sum_{n=1}^{\infty} f(n)$. The function f is clearly both **positive** and **decreasing**, so by the integral test the series converges iff $\int_2^{\infty} f(x)\,\mathrm{d}x$ converges. We consider

$$\int_2^{\infty} \frac{\mathrm{d}x}{x(\log x)^p} .$$

Susbtituting $u = \log x$ we have $\frac{\mathrm{d}x}{x} = \mathrm{d}u$ and $u \to \infty$as $x \to \infty$ so we have

$$\int_2^{\infty} \frac{\mathrm{d}x}{x(\log x)^p} = \int_2^{\infty} \frac{\mathrm{d}u}{u^p}$$

which converges when $p > 1$ and diverges otherwise. By the integral test the same holds for our series.

(c) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Solution: Let $f(x) = \frac{1}{1+x^2}$ which is clearly **positive** and **decreasing**. By the integral test the series $\sum_{n=1}^{\infty} f(n)$ converges iff the integral $\int_1^{\infty} \frac{\mathrm{d}x}{1+x^2}$ does. But

$$\int_1^{\infty} \frac{\mathrm{d}x}{1+x^2} = \lim_{T\to\infty} (\arctan(T) - \arctan(1)) = \lim_{T\to\infty} \arctan(T) - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4},$$

so the integtral and the series converge.

Solution: Let $f(x) = \frac{1}{1+x^2}$ which is clearly **positive** and **decreasing**. By the integral test the series $\sum_{n=1}^{\infty} f(n)$ converges iff the integral $\int_0^{\infty} \frac{\mathrm{d}x}{1+x^2}$ does. Converges does not depend on the starting point so we consider $\int_1^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x$. Now $\frac{1}{1+x^2} < \frac{1}{x^2}$ and $\int_1^{\infty} \frac{\mathrm{d}x}{x^2}$ converges by the p-test $(2 > 1)$ so $\int_1^{\infty} \frac{\mathrm{d}x}{1+x^2}$ converges by the comparison test, and $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by the integral test.

(2) The integral $\int_2^{\infty} \frac{x+\sin x}{1+x^2}\,\mathrm{d}x$ diverges. Why can't we use the integral test to conlcude that $\sum_{n=2}^{\infty} \frac{n+\sin n}{1+n^2}$ diverges as well?

Solution: The function $f(x) = \frac{x+\sin x}{1+x^2}$ isn't monotone:

$$\begin{aligned} f'(x) &= \frac{(1+\cos x)(1+x^2) - 2x(x+\sin x)}{(1+x^2)^2} \\ &= \frac{(1+\cos x - 2)\,x^2 - 2x\sin x + 1 + \cos x}{(1+x^2)^2} \\ &= \frac{(\cos x - 1)x^2 - 2x\sin x + 1 + \cos x}{(1+x)^2} . \end{aligned}$$

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In particular, if $x = 2\pi k$ ($k \in \mathbb{Z}$) then $\cos x = 1$, $\sin x = 0$ and

$$f'(x) = \frac{2}{(1+x^2)} > 0\,.$$

We'll later show that this series diverges.

2. Tail estimates (<u>**not examinable in Math 101**</u>)

(3) Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(a) Show that $\sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \frac{1}{N}$.
Solution: The fucntion $f(x) = \frac{1}{x^2}$ is decreasing and positive. By the integral test, $\sum_{n=N+1}^{\infty} f(n) \leq \int_N^{\infty} f(x)\,\mathrm{d}x = \left[-\frac{1}{x}\right]_N^{\infty} = \frac{1}{N}$.

(b) How many terms to we need to include to approximate the sum of the series within 10^{-5}?
Solution: We have $\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{N} \frac{1}{n^2} + \sum_{n=N+1}^{\infty} \frac{1}{n^2}$. If $N = 10^5$ we see that

$$0 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{10^5} \frac{1}{n^2} \leq 10^{-5}\,.$$

(3) (The harmonic series)

(a) Show that $\sum_{n=1}^{N} \frac{1}{n} \geq \log(N+1)$
Solution: $\sum_{n=1}^{N} \frac{1}{n} \geq \int_1^{N+1} \frac{\mathrm{d}x}{x} = \log(N+1)$.

(b) Show that $\sum_{n=1}^{N} \frac{1}{n} \leq (1 - \log 2) + \log(N+1)$
Solution: $\sum_{n=1}^{N} \frac{1}{n} \leq 1 + \int_2^{N+1} \frac{\mathrm{d}x}{x} = 1 + \log(N+1) - \log 2$.

(4) Bonus problem: $\gamma = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log(N+1)\right)$ exists.

(a) For $N \geq 1$ set $s_N = \sum_{n=1}^{N} \frac{1}{n} - \log(N+1)$ (set $s_0 = 0$) and let $a_n = s_n - s_{n-1}$. Show that $a_n = \frac{1}{n} - \log\left(1 + \frac{1}{n}\right)$.
Solution: We calculate:

$$\begin{aligned} s_N - s_{N-1} &= \left(\sum_{n=1}^{N} \frac{1}{n} - \log(N+1)\right) - \left(\sum_{n=1}^{N-1} \frac{1}{n} - \log(N)\right) \\ &= \left(\sum_{n=1}^{N} \frac{1}{n} - \sum_{n=1}^{N-1} \frac{1}{n}\right) - (\log(N+1) - \log(N)) \\ &= \frac{1}{N} - \log\left(\frac{N+1}{N}\right) \\ &= \frac{1}{N} - \log\left(1 + \frac{1}{N}\right)\,. \end{aligned}$$

(b) Show that there is $C > 0$ such that $0 \leq a_n \leq \frac{C}{n^2}$ for all $n \geq 1$. By the comparison test, $\sum_{n=1}^{\infty} a_n$ converges.
Solution: The function $f(x) = \log(1+x)$ is differentiable; we have $f'(x) = \frac{1}{1+x}$, $f''(x) = -\frac{1}{(1+x)^2}$, $f^{(3)}(x) = \frac{2}{(1+x)^3}$. Thus $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$ and hence for $x \geq 0$ we have

$$f(x) = x - \frac{1}{2}x^2 + \frac{1}{3}\frac{x^3}{(1+\xi)^3}$$

for some $\xi \in (0, x)$. For $0 \leq x \leq 1$ we see that $\xi \geq 0$ and hence

$$0 \leq \frac{1}{3}\frac{x^3}{(1+\xi)^3} \leq \left(\frac{x}{3(1+\xi)^3}\right)x^2 \leq \frac{1}{3}x^2\,.$$

It follows that for $0 \leq x \leq 1$ we have

$$x - \frac{1}{2}x^2 \leq \log(1+x) \leq x - \frac{1}{2}x^2 + \frac{1}{3}x^2$$

and hence

$$\frac{1}{6}x^2 \leq x - \log(1+x) \leq \frac{1}{2}x^2 \,.$$

Plugging in $x = \frac{1}{n}$ gives the claim.

(c) Show that $s_N = \sum_{n=1}^{N} a_n$. It follows that $\{s_N\}_{n=1}^{\infty}$ converges.

Solution: This is a telescoping series: $\sum_{n=1}^{N} a_n = (s_1 - s_0) + (s_2 - s_1) + \cdots + (s_N - s_{N-1}) = s_N - s_0 = s_N$.

The number γ is called the Euler–Mascheroni constant, its value is about 0.577.