Math 101 – SOLUTIONS TO WORKSHEET 31
MANIPULATING POWER SERIES

1. Manipulating power series: Geometric Series

Recall that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

(1) Find a power series representation for

(a) (Final 2014) $\frac{x^3}{1-x}$

Solution: We know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Multiplying by x^3 we find

$$\frac{x^3}{1-x} = x^3 \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+3} = \sum_{m=3}^{\infty} x^m\,.$$

(b) (Final 2011) $\frac{1}{1+x^3}$

Solution: We know that $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$. Substituting $u = -x^3$ we therefore get

$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty} \left(-x^3\right)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n}\,.$$

(2) Find a power series representation for $\frac{1}{x+3}$

(a) Expanding about $a = 0$

Solution: Striving toward $\frac{1}{1-u}$, we have:

$$\begin{aligned} \frac{1}{x+3} &= \frac{1}{3} \cdot \frac{1}{1+\left(\frac{x}{3}\right)} = \frac{1}{3} \cdot \frac{1}{1-\left(-\frac{x}{3}\right)} \\ &= \frac{1}{3} \sum_{n=0}^{\infty} \left(-\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^n\,. \end{aligned}$$

(b) Expanding about $a = 7$

Solution: We need $x - 7$ in our function, so we have:

$$\begin{aligned} \frac{1}{x+3} &= \frac{1}{x-7+10} = \frac{1}{10} \cdot \frac{1}{1-\left(-\frac{x-7}{10}\right)} \\ &= \frac{1}{10} \sum_{n=0}^{\infty} \left(-\frac{x-7}{10}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{10^{n+1}} (x-7)^n\,. \end{aligned}$$

2. Manipulating power series: Calculus

(3) (Final 2011) Evaluate the following indefinite integral as a power series, and find its radius of convergence: $\int \frac{\mathrm{d}x}{1+x^3}$

Solution: In 1(b) we found that

$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-1)^n x^{3n}\,.$$

Integrating term-by-term we find

$$\int \frac{\mathrm{d}x}{1+x^3} = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1} x^{3n+1}\,.$$

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The expansion $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$ has radius of convergence 1 (open interval $|u| < 1$). So the open interval for the expansion of $\frac{1}{1+x^3}$ was where $\left|-x^3\right| < 1$, that is where $|x|^3 < 1$, that is where $|x| < 1$, so the radius of convergence was 1. Since integration doesn't change the radius, the radius of convergence is still 1.

(4) Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $g(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$. Last time we verified that f converges everywhere, while g converges for $-1 < x \leq 1$.

(a) Find the power series representation of $f'(x)$. What is $f(x)$?
Solution: $f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{x^m}{m!} = f(x)$ so $f'(x) = f(x)$ and $f(x) = Ce^x$. Since $f(0) = 1$, we have $C = 1$ and $f(x) = e^x$.

(b) Find the power series representation of $g'(x)$. What is $g'(x)$? What is $g(x)$?
Solution: $g'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n x^{n-1}}{n} = \sum_{n=1}^{\infty} (-x)^{n-1} = \sum_{m=0}^{\infty} (-x)^m = \frac{1}{1-(-x)} = \frac{1}{1+x}$ so $g'(x) = \frac{1}{1+x}$ and $g(x) = \log(1+x) + C$. Since $g(0) = 0$, we have $C = 0$ and $g(x) = \log x$.

(c) Conclude that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log 2$.
Solution: Since $x = 1$ is in the domain of convergence of g, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} 1^n = g(1) = \log(1+1) = \log 2\,.$$

(d) Find the power series representation of $\int_0^x \exp(-t^2)\,\mathrm{d}t$.
Solution: We have $\exp(-t^2) = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}$. Integrating term-by-term we have

$$\int_0^x f(-t^2)\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{t^{2n+1}}{2n+1}\right]_{t=0}^{t=x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} t^{2n+1}\,.$$

3. Manipulating power series: summing series

(5) Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.
Solution: Let $h(x) = \sum_{n=1}^{\infty} nx^n$. We see that

$$\begin{aligned} h(x) &= x\sum_{n=1}^{\infty} nx^{n-1} = x\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=1}^{\infty} x^n \\ &= x\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty} x^n = x\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} \\ &= \frac{x}{(1-x)^2}\,. \end{aligned}$$

Now the radius of convergence of $\sum_{n=0}^{\infty} x^n$ is 1, so $\frac{1}{2}$ is in the domain of convergence and we conclude that

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = \frac{4}{2} = 2\,.$$