Math 101 – SOLUTIONS TO WORKSHEET 32
MANIPULATING POWER SERIES

1. Manipulating power series: Calculus

(1) Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $g(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$. We know that f converges everywhere, while g converges in $(-1, 1]$.

(a) Find the power series representation of $f'(x)$. What is $f(x)$?

Solution: $f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{x^m}{m!} = f(x)$ so $f'(x) = f(x)$ and $f(x) = Ce^x$. Since $f(0) = 1$, we have $C = 1$ and $f(x) = e^x$.

(b) Find the power series representation of $g'(x)$. What is $g'(x)$? What is $g(x)$?

Solution: $g'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} nx^{n-1}}{n} = \sum_{n=1}^{\infty} (-x)^{n-1} = \sum_{m=0}^{\infty} (-x)^m = \frac{1}{1-(-x)} = \frac{1}{1+x}$ so $g'(x) = \frac{1}{1+x}$ and $g(x) = \log(1 + x) + C$. Since $g(0) = 0$, we have $C = 0$ and $g(x) = \log x$.

(c) Conclude that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log 2$.

Solution: Since $x = 1$ is in the domain of convergence of g, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} 1^n = g(1) = \log(1 + 1) = \log 2\,.$$

(2) Consider the *error function* $\mathrm{erf}(x) = \int_0^x \exp(-t^2)\,\mathrm{d}t$.

(a) Find the power series expansion of $\mathrm{erf}(x)$ about zero.

Solution: We have $\exp(-t^2) = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}$. Integrating term-by-term we have

$$\int_0^x f(-t^2)\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{t^{2n+1}}{2n+1}\right]_{t=0}^{t=x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} t^{2n+1}\,.$$

(b) How many terms in the expansion are necessary to estimate $\mathrm{erf}(\frac{1}{2})$ to within 0.001?

Solution: We need to estimate $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)2^n}$ which is an alternating series. The term with $n = 4$ already has the magnitude

$$\frac{1}{24 \cdot 9 \cdot 16} < \frac{1}{20 \cdot 15 \cdot 8} = \frac{1}{2400} < \frac{1}{1000}$$

so taking the first four terms ($0 \le n \le 3$) suffices.

2. Manipulating power series: summing series

(3) Find $\sum_{n=1}^{\infty} \frac{1}{n2^n}$.

Solution: We know that $\log(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$, with radius of convergence 1. We then have:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n2^n} &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{-1}{2}\right)^n = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(-\frac{1}{2}\right)^n \\ &= = -\log\left(1 - \frac{1}{2}\right) = -\log\frac{1}{2} = \log 2\,.\end{aligned}$$

(4) Avatars of geometric series.

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(a) Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.
Solution: Let $h(x) = \sum_{n=1}^{\infty} nx^n$. We see that

$$\begin{aligned} h(x) &= x\sum_{n=1}^{\infty} nx^{n-1} = x\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=1}^{\infty} x^n \\ &= x\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty} x^n = x\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} \\ &= \frac{x}{(1-x)^2}\,. \end{aligned}$$

Now the radius of convergence of $\sum_{n=0}^{\infty} x^n$ is 1, so $\frac{1}{2}$ is in the domain of convergence and we conclude

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = \frac{4}{2} = 2\,.$$

(b) Express $\sum_{n=1}^{\infty} n^2 x^n$ as a *rational function* (ratio of polynomials).
Solution: Let $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. We see that

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} nx^{n-1} \\ xf'(x) &= \sum_{n=0}^{\infty} nx^n \\ (xf'(x))' &= \sum_{n=0}^{\infty} n^2x^{n-1} \\ x\,(xf'(x)) &= \sum_{n=0}^{\infty} n^2x^n\,, \end{aligned}$$

so that

$$\begin{aligned} \sum_{n=1}^{\infty} n^2x^n &= \sum_{n=0}^{\infty} n^2x^n = x\left(x\left(\frac{1}{1-x}\right)'\right)' = x\left(\frac{x}{(1-x)^2}\right)' \\ &= x\left(\frac{1}{(1-x)^2} + \frac{2x}{(1-x)^3}\right) = \frac{x\,((1-x)+2x)}{(1-x)^3} = \boxed{\frac{x(1+x)}{(1-x)^3}}\,. \end{aligned}$$

(5) Find a simple formula for $\sum_{n=0}^{\infty} \frac{e^{nx}}{n!}$.
Solution: We know that $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$ so setting $u = e^x$ we get $\sum_{n=0}^{\infty} \frac{1}{n!}e^{nx} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(e^x\right)^n = e^{e^x}$.