# Math 101 – SOLUTIONS TO WORKSHEET 34
# TAYLOR SERIES AND LIMITS

## 1. Derivatives

(1) (Final 2014) Let $\sum_{n=0}^{\infty} c_n x^n$ be the MacLaurin series for $e^{3x}$. Find $c_5$.
**Solution:** Knowing that $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$ we have $e^{3x} = \sum_{n=0}^{\infty} \frac{3^n}{n!} x^n$ so $c_5 = \frac{3^5}{5!}$.

(2) (Final 2013) Let $f(x) = x^2 \sin(x^3)$. Find $f^{11}(0)$.
**Solution:** We know that $\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots$ so

$$x^2 \sin(x^3) = x^2 \left( x^3 - \frac{x^9}{3!} + \cdots \right) = x^5 - \frac{x^{11}}{3!} + \cdots .$$

It follows that $\frac{f^{(11)}(0)}{11!} = \frac{1}{3!}$ so $f^{(11)}(0) = \frac{11!}{3!}$.

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(3) Let $g(x) = \begin{cases} \frac{e^{-x^2}-1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$.

(a) Find $g^{(3)}(0)$.
**Solution:** Knowing that $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$, the series for $e^{-x^2}$ begins $1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots$. We have

$$g(x) = \frac{-x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots}{x} = -x + \frac{1}{2}x^3 - \frac{1}{6}x^5 + \cdots .$$

By the formula for Taylor coefficients we have $\frac{g^{(3)}(0)}{3!} = \frac{1}{2}$ so $g^{(3)}(0) = 3$.

(b) (2011 Final) Give the first three non-zero terms of the MacLaurin series for $\int g(x)\,\mathrm{d}x$.
**Solution:** Integrating term-by-term, the expansion of $\int g\,\mathrm{d}x$ begins

$$C - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{36}x^6 + \cdots$$

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## 2. Limits without l'Hôpital's rule

(4) (Final 2012) Evaluate $\lim_{x\to 0} \frac{\sin(x) - x + x^3/6}{\sin(x^5)}$
**Solution:** We have $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots$ so $\sin x - x + \frac{1}{6}x^3 = \frac{1}{120}x^5 + \cdots$. From this we also get $\sin(x^5) = x^5 - \frac{1}{6}x^{15} + \cdots$. In summary we have:

$$\frac{\sin(x) - x + x^3/6}{\sin(x^5)} \approx \frac{\frac{1}{120}x^5}{x^5} = \frac{1}{120}$$

and hence

$$\lim_{x\to 0} \frac{\sin(x) - x + x^3/6}{\sin(x^5)} = \frac{1}{120} .$$

(5) Evaluate $\lim_{x\to 0} \frac{x \sin x - \log(1+x^2)}{e^{-x^2/2} - \cos(x)}$
**Solution:** We have $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots$ so $x \sin x \approx x^2 - \frac{1}{6}x^4$ to fourth order. Similarly, $\log(1+u) = u - \frac{u^2}{2} + \cdots$ so $\log(1+x^2) \approx x^2 - \frac{1}{2}x^4$ to fourth order. In the denominator we have

*Date*: 31/3/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

$e^u = 1 + u + \frac{1}{2}u^2 + \cdots$ so $e^{-\frac{1}{2}x^2} \approx 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4$ to fourth order. We have $\cos x \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ to fourth order. Correct to fourth order we therefore have

$$\frac{x \sin x - \log(1+x^2)}{e^{-x^2/2} - \cos(x)} \approx \frac{\left(x^2 - \frac{1}{6}x^4\right) - \left(x^2 - \frac{1}{2}x^4\right)}{\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) - \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right)} = \frac{\left(\frac{1}{2} - \frac{1}{6}\right)x^4}{\left(\frac{1}{8} - \frac{1}{24}\right)x^4} = \frac{\frac{1}{2}\left(1 - \frac{1}{3}\right)}{\frac{1}{8}\left(1 - \frac{1}{3}\right)} = 4$$

so that

$$\lim_{x \to 0} \frac{x \sin x - \log(1+x^2)}{e^{-x^2/2} - \cos(x)} = 4\,.$$