Math 538: Problem Set 3

Review: Ideals and primes

Fix a ring R

1. (The product operation on ideals)
   (a) Show that the operation of multiplication of ideals in R is commutative and associative, and has the identity element R. If R is an integral domain show that $IJ \neq (0)$ unless $I = (0)$ or $J = (0)$.
   (b) Now let P, I, J be ideals of R with P prime and $P \supset IJ$. Show that $P \supset I$ or $P \supset J$.
   (c) Extend (b) to the case of $P \supset \prod_{i=1}^{n} I_i$ where I_i is a finite set of ideals.

2. (The CRT)
   (a) Let $I_1, \ldots, I_k$ be ideals, and suppose that $I_i + I_j = R$ where $i \neq j$. Show that $\prod_{i=1}^{k} I_i = \bigcap_{i=1}^{k} I_i$ and that the natural map $R / \cap_{i=1}^{k} I_i \to \oplus_{i=1}^{k} R/I_i$ is a well-defined isomorphism of rings.
   (b) Suppose that $\sum_{i=1}^{k} I_i = R$, and let $n_i \in \mathbb{Z}_{\geq 1}$. Show that $\sum_{i=1}^{k} I_i^{n_i} = R$.

Unique factorization

Let L/K be an extension of number fields.

3. (Primes above and below)
   (a) Let $\mathfrak{A} \lhd \mathcal{O}_L$ be a non-zero proper ideal. Show that $\mathfrak{A} \cap K = \mathfrak{A} \cap \mathcal{O}_K$ and that this is a non-zero proper ideal of $\mathcal{O}_K$.
   (b) Let $\mathfrak{P} \lhd \mathcal{O}_L$ be a non-zero prime ideal. Show that $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_K$ is a prime ideal of $\mathcal{O}_K$. We say that $\mathfrak{P}$ *lies above* $\mathfrak{p}$ and write $\mathfrak{P} | \mathfrak{p}$.
   (c) Show that $\mathfrak{P}$lies above $\mathfrak{p}$ iff $\mathfrak{P} | \mathfrak{p}\mathcal{O}_L$ as ideals of $\mathcal{O}_L$.

4. The map $\mathfrak{a} \mapsto \mathfrak{a}\mathcal{O}_K$.
   (a) Let $\mathfrak{a} \lhd \mathcal{O}_K$ be a proper ideal. Show that there is $\gamma \in K \setminus \mathcal{O}_K$ such that $\gamma\mathfrak{a} \subset \mathcal{O}_K$, and conclude that $\mathfrak{a}\mathcal{O}_L$ is a proper ideal of $\mathcal{O}_L$.
   (b) Let $\mathfrak{a}, \mathfrak{b}$ be (fractional) ideals of $\mathcal{O}_K$. Show that $(\mathfrak{a}\mathfrak{b})\mathcal{O}_L = (\mathfrak{a}\mathcal{O}_L)(\mathfrak{b}\mathcal{O}_L)$.
   (c) Let $\mathfrak{a}, \mathfrak{b}$ be ideals of $\mathcal{O}_K$. Comparing prime factorizations in $\mathcal{O}_K, \mathcal{O}_L$ show that $\mathfrak{a}\mathcal{O}_L | \mathfrak{b}\mathcal{O}_L \Rightarrow \mathfrak{a}|\mathfrak{b}$. Conclude that the map $\mathfrak{a} \to \mathfrak{a}\mathcal{O}_L$ is injective on fractional ideals.
   (d) Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$. Show that $\mathfrak{a}\mathcal{O}_L \cap \mathcal{O}_K = \mathfrak{a}$.

Localization

5. (Localization at a prime) Let $\mathfrak{p} \lhd \mathcal{O}_K$ be a prime of K.
   (a) Show that $\mathcal{O}_{K,\mathfrak{p}} = \left\{ \frac{\alpha}{s} \mid \alpha, s \in \mathcal{O}_K, s \notin \mathfrak{p} \right\}$ is a subring of K. It is called the *localization* of $\mathcal{O}_K$ at $\mathfrak{p}$.
   (b) Show that $\mathfrak{p}\mathcal{O}_{K,\mathfrak{p}}$ is an ideal in $\mathcal{O}_{K,\mathfrak{p}}$, and that its complement consists of $(\mathcal{O}_{K,\mathfrak{p}})^{\times}$. Conclude that $\mathcal{O}_{K,\mathfrak{p}}$ is a *local ring*: it has a unique maximal ideal.
   RMK The localization of any ring at a prime ideal is a local ring.
   (c) Let $x \in K^{\times}$. By considering the prime factorization of the fractional ideal (x) show that at least one of $x, x^{-1} \in \mathcal{O}_{K,\mathfrak{p}}$.
   DEF A subring of a field satisfying the property of (c) is called a *valuation ring.*
   (**d) Show that every ideal of $\mathcal{O}_{K,\mathfrak{p}}$ is of the form $(\mathfrak{p}\mathcal{O}_{K,\mathfrak{p}})^k$ for some $k \geq 0$.

(e) Let L be a finite extension of K. Show that $\left\{\frac{x}{s} \mid x \in \mathcal{O}_L, s \in \mathcal{O}_K \setminus \mathfrak{p}\right\}$ is a subring of L (the localization of $\mathcal{O}_L$ at $\mathfrak{p}$), finitely generated as an $\mathcal{O}_{K,\mathfrak{p}}$-module, and use the structure theory of modules over a PID to conclude that it is of the form $\mathcal{O}_{K,\mathfrak{p}}^n$ where $n = [L : K]$.

Completion

6. Let (X,d) be a metric space, $\left(\hat{X},\hat{d}\right)$ its metric completion. Let (Y,d_Y) be a complete metric space.
   (a) Let $f\colon X^n \to Y$ be *uniformly continuous on balls*: Given $z \in X$ and $\varepsilon, R > 0$ there is $\delta > 0$ such that if $\underline{x}, \underline{x}' \in \mathbb{R}^n$ satisfy for all i that $d(x_i, x_i') < \delta$ for $d(z,x_i), d(z,x_i') \leq R$ then
   $$d_Y\left(f(\underline{x}), f(\underline{x}')\right) < \varepsilon\,.$$
   Show that f extends uniquely to a continuous function $\hat{f}\colon \hat{X}^n \to Y$.
   (b) Suppse that X is also a group, and that the map $(x,x') \to x^{-1}x'$ is uniformly continuous. Show that $\hat{X}$ has a unique group structure continuosly extending that of X.
   (c) In the setting of (b), let $H < X$ be a subgroup. Show that the closure of H in $\hat{X}$ is a subgroup, and that if H is normal in X then the closure is normal in $\hat{X}$.

7. (Completion of a ring) Let R be a ring, $I \lhd R$ an ideal. Suppose that $\bigcap_{n\geq 1} I^n = (0)$. Let $\hat{R}_I = \left\{(x_n) \in \prod_{n\geq 1} R/I^n \mid \forall n \geq m : (x_n - x_m) \in I^m\right\}$, and let $\pi_n\colon \hat{R}_I \to R/I^n$ be the projection on the nth coordinate.
   (a) For $n \geq 2$, let $f_n\colon R/I^n \to R/I^{n-1}$ be the natural quotient map. Show that $f_{n+1} \circ \pi_{n+1} = \pi_n$ for all $n \geq 1$.
   (b) Let S be a ring equipped with a system of ring homomorphisms $\varphi_n\colon S \to R/I^n$ such that $f_{n+1} \circ \varphi_{n+1} = \varphi_n$. Show that there exists a unique homomorphism $F\colon S \to \hat{R}_I$ such that $\varphi_n = \pi_n \circ F$ for all n.
   (c) Show that a pair $\left(\hat{R}_I, (\pi_n)_{n=1}^{\infty}\right)$ satisfying (a),(b) is unique up to a unique isomorphism. It is called the *completion* of R at the ideal I.
   (d) Show that the "diagonal map" $\Delta\colon R \to \hat{R}_I$ where $\pi_n \circ \Delta$ is the quotient map $R \to R/I^n$ is an injective ring homomorphism. Conclude that π_n is surjective.
   (e) Show that $\hat{R}_I$ equipped with the metric $d\left((x_n),(y_n)\right) = e^{-\min\{n \mid x_n \neq y_n\}}$ is the metric completion of R with respect to the metric $d(x,y) = e^{-\max\{n \mid (x-y) \in I^n\}}$. Here we take the conventions that $I^0 = R$, that the minimum of the empty set and the maximu of $\mathbb{N}$ are infinity and that $e^{-\infty} = 0$.