# Math 538: Problem Set 4 (due 24/3/2017)

Do a good amount of problems; choose problems based on what you already know and what you need to practice. Problems 2, 4, 6-8. 3 is useful for perspective.

## Valuations and absolute values

1. Let $R$ be an integral domain, $K(R)$ its field of fractions.
   (a) Show that any absolute value on $K(R)$ is uniquely determined by its restriction to $R$.
   (b) Let $|\cdot| : R \to \mathbb{R}_{\geq 0}$ be a map satisfying the definition of absolute values restricted to $R$. Show that it extends to an absolute value of $K(R)$.

2. Prove *Ostrowski's Theorem*: every absolute value on $\mathbb{Q}$ either trivial, equivalent to $|\cdot|_\infty$ or equivalent to $|\cdot|_p$ for some $p$.

3. (Function fields) Let $F$ be a field, $v$ a valuation on $F(t)$ which is trivial on $F$ (for example, $F$ might be a finite field).
   (a) Suppose that $v(f) < 0$ for some $f \in F[t]$. Show that $v(t) < 0$ and that up to rescaling $v = v_\infty$ where $v_\infty(f) = -\deg f$.
   (b) Suppose that $v(f) > 0$ for some $f \in F[t]$. Show there is an irreducible $p \in F[t]$ with that property, and that up to rescaling $v = v_p$ where $v_p(p^r \frac{f}{g}) = r$ where $(p, fg) = 1$.
   (c) (Product formula) Fix $q > 1$ and set $|f|_\infty = q^{\deg f}$ for $f \in F[t]$. For each irreducible monic $p \in F[t]$ normalize the corresponding absolute value by $|f|_p = q^{-(\deg p)v_p(f)}$. Show that for all $f \in \mathbb{Q}(t)^\times$,
   $$|f|_\infty \cdot \prod_{p \text{ monic irred}} |f|_p = 1.$$
   (d) Fix a transcendental $\alpha \in \mathbb{R}$ (or $\alpha \in \mathbb{Q}_p$) and for $f \in \mathbb{Q}(t)$ set $|f| = |f(\alpha)|_\infty$ (or $|f| = |f(\alpha)|_p$). Show that this defines an absolute value on $\mathbb{Q}(t)$ which is not of the forms found in parts (a),(b).

4. Let $(X,d)$ be an ultrametric space ($\forall x, y, z : d(x,z) \leq \max\{d(x,y), d(y,z)\}$)
   (a) ("all triangles are isoceles") Let $x, y, z \in X$ and suppose that $d(y,z) < d(x,y)$. Show that $d(x,z) = d(x,y)$.
   (b) Let $B_1, B_2$ be two closed balls in $X$. Show that one of the following holds: $B_1 \subset B_2$, $B_2 \subset B_1$, $B_1 \cap B_2 = \emptyset$.
   (c) Show that the completion of an ultrametric space is ultrametric.
   (d) Suppose that $X$ is complete, and let $\{x_i\}_{i=1}^\infty \subset X$. Show that the sequence converges iff $\lim_{i \to \infty} d(x_i, x_{i+1}) = 0$.
   (e) (Calculus student's dream) Let $\{a_n\}_{n=0}^\infty \subset \mathbb{Q}_p$. Show that $\sum_{n=0}^\infty a_n$ converges iff $\lim_{n \to \infty} a_n = 0$.

5. (Odds and ends)
   (a) Find the $a_j \in \{0, \ldots, p-1\}$ such that $-1 = \sum_{j=0}^{\infty} a_j p^j$ in $\mathbb{Z}_p$.
   (b) Show that every $n \in \mathbb{Z}_{\geq 0}$ has a unique representation in the form $\sum_{j=0}^{J} a_j p^j$ (finite sum), $a_j \in \{0, \ldots, p-1\}$
   (c) Suppose $p$ is odd. Show that every $n \in \mathbb{Z}$ has a unique representation in the form $\sum_{j=0}^{J} b_j p^j$ (finite sum), $b_j \in \left\{ -\frac{p-1}{2}, \ldots, \frac{p-1}{2} \right\}$.
   (d) (Moving to $\mathbb{Q}_p$ is equivalent to localization) Show that $\mathbb{Q} \cap \mathbb{Z}_p = \mathbb{Z}_{(p)} = \left\{ \frac{x}{s} \mid x, s \in \mathbb{Z}, p \nmid s \right\}$.

6. Show that $\mathbb{Z}_p$, hence $\mathbb{Q}_p$ has the cardinality of the continuum.

## Structure theory of $\mathbb{Q}_p$

Fix a field $F$ and a non-archimedean absolute value $|\cdot|$ on $F$.

7. (Basic structure)
   (a) Let $R = \{x \in F \mid |x| \leq 1\}$. Show that $R$ is a subring of $F$, and that it is a *valuation ring*: for all $x \in F^\times$ at least one of $x, x^{-1}$ is in $R$. In particular, $F$ is the field of fractions of $R$.
   (b) Show that $R^\times = \{x \in F \mid |x| = 1\}$.
   (c) (Analogue of "ring of integers") Show that $R$ is *integrally closed*: if $\alpha \in F$ is a root of a monic $f \in R[x]$ then $\alpha \in R$ (hint: supposing $|\alpha| > 1$ calculate $|f(\alpha)|$ using 3(a)).
   (d) Show that $P = \{x \in R \mid |x| < 1\}$ is the unique maximal ideal of $R$ and that it is open in $R$. In particular, $R/P$ is a field and its natural topology is discrete.

8. (Discrete valuations) Suppose now that $\{|x| \mid x \in F^\times\}$ is a discrete subset of $\mathbb{R}_{>0}^\times$ (equivalently that $\{\log |x| \mid x \in F^\times\}$ is a discrete subgroup of $\mathbb{R}$).
   (a) Show that $\log |F^\times| = r\mathbb{Z}$ for some $r > 0$.
   (b) Show that $|\cdot|$ is non-archimedean.
   (c) ("Unique factorization") Let $\varpi \in P$ (read "pi", not "omega") have absolute value $e^{-r}$ exactly. Show that the ideals of $R$ are exactly $(\varpi^k)$ for $k \geq 0$ and $(0)$.
   (d) ("Onion rings") Show that $\{x : |x| = e^{-kr}\} = \varpi^k R^\times$.
   (e) (Multiplicative group) For $k \geq 1$ show that $U_k = 1 + \varpi^k R$ is an open (hence also closed) subgroup of $U = R^\times$. Show that $U/U_1 \simeq (R/P)^\times$ and that for $k \geq 1$ $U_k/U_{k+1} \simeq (R/P, +)$.
   (f) ("Passing to the completion merely reorganizes the information") Compare (c)-(e) with the corresponding results for $\mathbb{Z}/p^K\mathbb{Z}$ we saw earlier.

9. (Representation of elements) In addition to the assumption of 7, suppose now that $F$ is complete with respect to the absolute value.
   (a) ("Power series") Let $A \subset R$ be a set of representatives for $R/P$, with $0 \in A$. Show that every element of $R$ has a unique representation in the form $\sum_{j=0}^{\infty} a_j \varpi^j$ where $a_j \in A$.
   (b) ("Laurent series") Show that every non-zero element of $F$ has a unique representation in the form $\sum_{j=J}^{\infty} a_j \varpi^j$ where $a_j \in A$, $J \in \mathbb{Z}$ and $a_J \neq 0$.
   (c) Show that $F$ is locally compact iff $R$ is compact iff $R^\times$ is compact iff $R/P$ is finite.

### Newton's method and Hensel's Lemma

10. Let $f: (a,b) \to \mathbb{R}$ be twice differentiable with $f'$ non-vanishing on the interval and set $F(x) = x - \frac{f(x)}{f'(x)}$.

   (a) Suppose $\left| \frac{f(\xi)f''(\xi)}{(f'(\xi))^2} \right| \leq c < 1$ for all $\xi \in (a,b)$. Show that $|F(x) - F(y)| \leq c\,|x-y|$ for all $x,y \in (a,b)$.

   (b) (Linear convergence) Suppose $z \in (a,b)$ has $f(z) = z$. Show that for any $x_0 \in (a,b)$ setting $x_{n+1} = F(x_n)$ has $x_n \in (a,b)$ for all $n$ and that $\lim_{n\to\infty} x_n = z$.

   (c) (Quadratic convergence) Show that there is $C$ such that for $x$ close enough to $z$, $|F(x) - z| \leq C\,|x-z|^2$.

11. Let $K$ be a field complete with repsect to a non-trivial non-archimedean absolute value and let $L$ be an algebraic extension of $K$. Show that there is a unique absolute value on $L$ extending that of $K$. (*Hint:* you need to cobble together results from class).