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ABSTRACT. In this note we prove that the Hasse-Weil zeta function of a curve is a rational function and satisfies a functional equation. We follow [Must, Chapter 3].

1. PRELIMINARIES AND NOTATION: QUICK REVIEW

Throughout this note X is a smooth projective curve over $k := \mathbb{F}_q$. A Weil divisor $D \in \mathrm{Div}(X)$ on X is a finite formal sum of the form

$$D = \sum_{x \in X_{\mathrm{cl}}} n_x x,$$

where X_{cl} are the closed points of X. We identify each closed point in X_{cl} with the orbit of a point in $X(\overline{\mathbb{F}_q})$ under the action of $\mathrm{Gal}(\overline{\mathbb{F}_q}|\mathbb{F}_q)$. The degree of a closed point $x \in X_{\mathrm{cl}}$ is $\deg(x) = [k(x) : k]$, where $k(x)$ is the residue field of x. The degree of the divisor D is

$$\deg(D) = \sum_{x \in X_{\mathrm{cl}}} n_x \deg(x).$$

Example 1.1. *Let $X = \mathbb{A}^1_{\mathbb{F}_3} = \mathrm{spec}(\mathbb{F}_3[x])$. Then $P = \mathrm{spec}\left(\mathbb{F}_3[x] \big/ (x^2+1)\right)$ is a closed point of X corresponding to the maximal ideal (x^2+1) of $\mathbb{F}_3[x]$. The residue field is $\mathbb{F}_3(P) = \mathbb{F}_3[x] \big/ (x^2+1)$ which is a degree 2 extension of $\mathbb{F}_3$. Hence the divisor $D = P$ has degree $\deg(D) = [\mathbb{F}_3(P) : \mathbb{F}_3] = 2$.*

Because X is a smooth projective curve we may identify a Weil divisor $D \in \mathrm{Div}(X)$ with its induced line bundle $\mathcal{L} = \mathcal{O}_X(D)$. We write $\deg(\mathcal{O}_X(D)) = \deg(D)$.

We say that two Weil divisors $D, D' \in \mathrm{Div}(X)$ are linearly equivalent and write $D \sim D'$ iff $D - D' = \mathrm{div}(f)$ for some $f \in k(X)^\times$. We write $\mathrm{Pic}(X)$ to denote the group of the divisors on X modulo this equivalence relation. Note that linearly equivalent Weil divisors correspond to isomorphic line bundles. In other words, $\mathrm{Pic}(X)$ is the group of line bundles on X modulo the isomorphism relation. We write $[D]$ for a divisor class in $\mathrm{Pic}(X)$.

Since our curve X is projective, linearly equivalent divisors have the same degree and hence the degree map descends to give a group homomorphism $\deg : \mathrm{Pic}(X) \to \mathbb{Z}$. The kernel of this homomorphism is denoted by $\mathrm{Pic}^0(X)$. We recall the Riemann-Roch theorem.

Theorem 1.2. *Let $D \in \mathrm{Div}(X)$ and write $\mathcal{K}$ for the canonical divisor of X. We have*

$$\ell(D) - \ell(\mathcal{K} - D) = \deg D - g + 1.$$

Moreover, $\deg(\mathcal{K}) = 2g-2$ *and*

$$\ell(D) = \deg(D) - g + 1, \text{ if } \deg(D) \geq 2g-1.$$

In the following we will make use of the following corollary of the Riemann-Roch.

Proposition 1.3. *The number of effective divisors in* $\mathrm{Div}(X)$ *that are linearly equivalent to* $D \in \mathrm{Div}(X)$ *is* $\frac{q^{\ell(D)}-1}{q-1}$. *If in particular* $\deg(D) \geq 2g-1$, *then the number of effective divisors in* $\mathrm{Div}(X)$ *that are linearly equivalent to* D *is* $\frac{q^{\deg D - g + 1}-1}{q-1}$.

Remark 1.4. *Recall that for* $D, D' \in \mathrm{Div}(X)$ *with* $D \sim D'$ *we have* $\ell(D) = \ell(D')$. *Therefore the integer* $\ell([D]) := \ell(D)$ *is well defined for a divisor class in* $\mathrm{Pic}(X)$.

2. Rationality

In this section we aim to prove the following strong form of the rationality conjecture in the setting of a smooth projective curve X over $\mathbb{F}_q$.

In the following we write

$$\mathrm{Pic}^0(X) = \{[D] \in \mathrm{Pic}(X) \ : \ \deg([D]) = 0\},$$

and

$$\mathrm{Pic}^m(X) = \{[D] \in \mathrm{Pic}(X) \ : \ \deg([D]) = m\},$$

To state the strong form of the rationality conjecture we aim to prove, we will first see that $\mathrm{Pic}^0(X)$ is a finite subgroup of $\mathrm{Pic}(X)$. We will write $h := |\mathrm{Pic}^0(X)|$.

Lemma 2.1. *We have that*

1. $\mathrm{Pic}^0(X)$ *is a finite subgroup of* $\mathrm{Pic}(X)$, *we write* $h := |\mathrm{Pic}^0(X)|$.
2. $\deg(\mathrm{Pic}(X)) = e\mathbb{Z}$ *for some* $e \in \mathbb{N}_{>0}$ *and if we write* $h := |\mathrm{Pic}^0(X)|$, *we have*

$$|\{[D] \in \mathrm{Pic}(X) \ : \ \deg([D]) = m\}| = \begin{cases} h & , e|m \\ 0 & , \text{otherwise.} \end{cases}$$

Proof. We will first prove the first part of this lemma. It is easy to see that $\mathrm{Pic}^0(X)$ is a group. We will prove that $\mathrm{Pic}^0(X)$ is finite. Let $D_n \in \mathrm{Div}(X)$ be such that $\deg(D_n) := n \geq 2g$. Notice that the map

$$\begin{aligned} \mathrm{Pic}^0(X) &\to \mathrm{Pic}^n(X) \\ [D] &\mapsto [D + D_n], \end{aligned}$$

gives a bijection between $\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^n(X)$. Therefore, it suffices to prove that $\mathrm{Pic}^n(X)$ is a finite set. We claim that for any divisor class $[D] \in \mathrm{Pic}^n(X)$, there exists an effective divisor $D' \in \mathrm{Div}(X)$ such that $[D] = [D']$. This is a consequence of the Riemann-Roch. Since $\deg(D) \geq 2g > 2g-1$, we have that $\ell(D) = n - g + 1 > 0$, hence D is linearly equivalent to an effective divisor D'. Thus it suffices to see that there is a finite number of effective divisors of degree n. This holds since there are only finitely many ways to write

n as a sum of positive numbers and there are only finitely many closed points in X_{cl} with degree less than n.

For the second part of this lemma, notice that $\deg(\text{Pic}(X))$ is an ideal of $\mathbb{Z}$, therefore it can be written as $\deg(\text{Pic}(X)) = e\mathbb{Z}$ for some $e \in \mathbb{N}_{>0}$. Fix a divisor class $[D_m] \in \text{Pic}^{em}(X)$. The map

$$\begin{aligned} \text{Pic}^0(X) &\to \text{Pic}^{em}(X) \\ [D] &\mapsto [D + D_m], \end{aligned}$$

gives a bijection between $\text{Pic}^0(X)$ and $\text{Pic}^{em}(X)$. The lemma follows. □

Theorem 2.2. *If X is a smooth projective curve over $\mathbb{F}_q$ of genus g such that X is irreducible over $\overline{\mathbb{F}_q}$, we have*

$$Z(X,t) = \frac{f(t)}{(1-t)(1-qt)},$$

where $f \in \mathbb{Z}[t]$ is a polynomial of degree $\deg(f) \leq 2g$, such that $f(0) = 1$ and $f(1) = h$.

We begin by establishing some key lemmas.

Lemma 2.3. *Let X be a variety over $\mathbb{F}_q$ and X' be the same variety over $\mathbb{F}_{q^r}$. Then*

$$Z(X', t^r) = \prod_{i=1}^{r} Z(X, \xi^i t),$$

where ξ is a primitive $r-$th root of unity.

Proof. Let $N_m = |X(\mathbb{F}_{q^m})|$ and $N'_m = |X'(\mathbb{F}_{q^{rm}})|$. We want to prove that

$$\exp\left(\sum_{m\geq 1} \frac{N'_m}{m} t^{rm}\right) = \prod_{i=1}^{r} \exp\left(\sum_{\ell \geq 1} \frac{N_\ell}{\ell} \xi^{\ell i} t^\ell\right),$$

or equivalently that

$$\sum_{m\geq 1} \frac{N'_m}{m} t^{rm} = \sum_{\ell\geq 1} \frac{N_\ell}{\ell} \left(\sum_{i=1}^{r} \xi^{\ell i}\right) t^\ell.$$

The desired equality follows from the fact that $N'_m = N_{rm}$ for all $m \geq 1$ and

$$\sum_{i=1}^{r} \xi^{\ell i} = \begin{cases} 0 & r \nmid \ell \\ r & \text{otherwise.} \end{cases}$$

□

Proof of Theorem 2.2. Last time we saw that

$$Z(X,t) = \sum_{D\geq 0} t^{\deg(D)}.$$

Denote by $a_{[D]} := |\{D' \in [D] \ : \ D' \geq 0\}|$. We may write

$$Z(X,t) = \sum_{[D]\in \mathrm{Pic}(X)} a_{[D]} t^{\deg([D])}.$$

We break this sum into two components depending on whether $\deg([D]) \geq 2g-1$ or $\deg([D]) \leq 2g-2$. Then

$$Z(X,t) = \sum_{[D]\in \mathrm{Pic}(X), \deg([D])\leq 2g-2} a_{[D]} t^{\deg([D])} + \sum_{[D]\in \mathrm{Pic}(X), \deg([D])\geq 2g-1} a_{[D]} t^{\deg([D])}. \tag{1}$$

We will now prove the first part of this theorem. That is that $Z(X,t)$ is a rational function. Notice that

$$S_1(t) := \sum_{[D]\in \mathrm{Pic}(X), \deg([D])\leq 2g-2} a_{[D]} t^{\deg([D])}, \tag{2}$$

is a polynomial. Therefore, it suffices to prove that

$$S_2(t) := \sum_{[D]\in \mathrm{Pic}(X), \deg([D])\geq 2g-1} a_{[D]} t^{\deg([D])}$$

is a rational function. By Proposition 1.3, we get

$$S_2(t) = \sum_{[D]\in \mathrm{Pic}(X), \deg([D])\geq 2g-1} \frac{q^{\deg([D])-g+1}-1}{q-1} t^{\deg([D])}.$$

Notice now that in view of Lemma 2.1 we have $\deg(\mathrm{Pic}(X)) = e\mathbb{Z}$ for some $e \in \mathbb{N}_{>0}$ and

$$|\{[D] \in \mathrm{Pic}(X) \ : \ \deg([D]) = m\}| = \begin{cases} h & e|m \\ 0 & \text{otherwise.} \end{cases}$$

Let d_0 be the smallest integer such that $d_0 e \geq 2g-1$. We have

$$S_2(t) = \sum_{d\geq d_0} h \frac{q^{de-g+1}-1}{q-1} t^{de} = \frac{h}{(q-1)} \left(q^{1-g} \cdot \frac{(qt)^{d_0 e}}{1-(qt)^e} - \frac{t^{d_0 e}}{1-t^e} \right). \tag{3}$$

This finishes the proof that $Z(X,t)$ is a rational function.

We proceed now to establish the complete statement of the theorem. Notice that $S_1(t) = g(t^e)$ for some polynomial $g \in \mathbb{Z}[t]$ with degree $\deg(g) \leq \frac{2g-2}{e}$. Combining this with (3) we get

$$Z(X,t) = \frac{f(t^e)}{(1-t^e)(1-q^e t^e)}, \tag{4}$$

where $f \in \mathbb{Q}[t]$ with $\deg(f) \leq \max\{2 + \frac{2g-2}{e}, d_0 + 1\}$. In fact since $Z(X,t) \in \mathbb{Z}[[t]]$ we see that $f \in \mathbb{Z}[t]$. We will now show that $e = 1$.

Note that the fact that $S_1(t)$ is a polynomial together with the expression in (3) yield

$$\lim_{t\to 1}(t-1)Z(X,t) = \lim_{t\to 1}\frac{-ht^{d_0 e}(t-1)}{q(1-t^e)}.$$

Therefore, $Z(X,t)$ has a pole of order 1 at $t=1$. If we now consider X' to be the curve X over $\mathbb{F}_{q^e}$, in view of Lemma 2.3, we get

$$Z(X',t^e) = \prod_{i=1}^{e} Z(X,\xi^i t),$$

for a $e-$th primitive root of unity ξ. This equation combined with (4) gives that

$$Z(X',t^e) = Z(X,t)^e.$$

However as we have seen $Z(X,t)$ as well as $Z(X',t)$ has a pole of order 1 at $t=1$, which gives $e=1$.

We have thus established that $e=1$.

Since $e=1$, we have $d_0 = 2g-1$. Therefore, the fact that $\deg(f) \le \max\{2+\frac{2g-2}{e}, d_0+1\}$ implies that $\deg(f) \le 2g$.

If in particular $g=0$ we have

$$Z(X,t) = \frac{h}{(1-t)(1-qt)}.$$

Finally, if $g \ge 1$ we have $f(0)=1$ and $f(1)=h$ as one can easily see from (2) and (3). □

In the course of the proof of Theorem 2.2 we saw that $\deg(\mathrm{Pic}(X)) = \mathbb{Z}$. Combining this with Lemma 2.1, we get the following corollary.

Corollary 2.4. *All* $\mathrm{Pic}^m(X)$ *have the same non-zero number of elements* $h = |\mathrm{Pic}^0(X)|$.

Before proceeding to prove the functional equation, we make some remarks on the existence of divisors with degree one.

Remark 2.5.

- If a curve X has an $\mathbb{F}_q$ point, then it has a divisor over $\mathbb{F}_q$ of degree 1. However, the converse is not true.
- If a curve defined over any field K, has genus 0 or 1 then it has an $K-$point if and only if it has a divisor over K of degree 1. This is a consequence of the Riemann-Roch which in this case implies that every divisor is linearly equivalent to an effective divisor.
- Corollary 2.4 is not true for smooth curves over a number field K. For example we consider the smooth conic $C\ :\ x^2+y^2+z^2=0$ over $\mathbb{Q}$. This conic has no divisor of degree 1. Indeed, if it had, since it has genus 0 it would also have a $\mathbb{Q}-$point. However C is pointless over $\mathbb{Q}$. In fact, one can see that since the canonical divisor of C has degree -2 we have $\deg(\mathrm{Pic}(C)) = 2\mathbb{Z}$.

- If the curve X has a divisor of degree n and m for two coprime integers n and m, then it has a divisor of degree 1. Most times (more specifically when the genus of the curve is not 1) we can find a divisor of even degree. This is the case because the canonical divisor on curve of genus g has degree $2g-2$. Thus if a curve of genus not equal to one has a divisor of odd degree, then it also has a divisor of degree 1.

3. Functional equation

We are now going to prove that the Hasse-Weil zeta function of a curve satisfies a functional equation, as stated in the theorem below.

Theorem 3.1. *If X is a smooth projective curve over $\mathbb{F}_q$ of genus g such that X is irreducible over $\overline{\mathbb{F}_q}$, we have*

$$Z\left(X, \frac{1}{qt}\right) = q^{1-g}t^{2-2g}Z(X,t).$$

Proof. As in the proof of Theorem 2.2 we write

$$\begin{aligned}
Z(X,t) &= \sum_{m=0}^{2g-2} \sum_{[D]\in \mathrm{Pic}^m(X)} \frac{q^{\ell([D])}-1}{q-1}t^m + \sum_{m\geq 2g-1} h\frac{q^{m-g+1}-1}{q-1}t^m \\
&= \sum_{m=0}^{2g-2} \sum_{[D]\in \mathrm{Pic}^m(X)} \frac{q^{\ell([D])}-1}{q-1}t^m + \frac{h}{(q-1)}\left(q^{1-g}\cdot\frac{(qt)^{2g-1}}{1-qt} - \frac{t^{2g-1}}{1-t}\right) \\
&= \sum_{m=0}^{2g-2} \sum_{[D]\in \mathrm{Pic}^m(X)} \frac{q^{\ell([D])}}{q-1}t^m - \sum_{m=0}^{2g-2}\frac{h}{q-1}t^m + \frac{h}{(q-1)}\left(q^{1-g}\cdot\frac{(qt)^{2g-1}}{1-qt} - \frac{t^{2g-1}}{1-t}\right) \\
&= \sum_{m=0}^{2g-2} \sum_{[D]\in \mathrm{Pic}^m(X)} \frac{q^{\ell([D])}}{q-1}t^m + \frac{h}{(q-1)}\left(q^{1-g}\cdot\frac{(qt)^{2g-1}}{1-qt} - \frac{1}{1-t}\right) := F(t)+G(t),
\end{aligned}$$

where $F(t) = \sum_{m=0}^{2g-2} \sum_{[D]\in \mathrm{Pic}^m(X)} \frac{q^{\ell([D])}}{q-1}t^m$ and $G(t) = \frac{h}{(q-1)}\left(q^{1-g}\cdot\frac{(qt)^{2g-1}}{1-qt} - \frac{1}{1-t}\right)$.

We now compute

$$\begin{aligned}
\frac{(q-1)}{h}G(1/qt) &= q^{1-g}\cdot\frac{t^{1-2g}}{1-t^{-1}} - \frac{1}{1-(qt)^{-1}} \\
&= \frac{q^{1-g}t^{2-2g}}{t-1} - \frac{qt}{qt-1} \\
&= t^{2-2g}q^{1-g}\left(\frac{q^g t^{2g-1}}{1-qt} - \frac{1}{t-1}\right) \\
&= t^{2-2g}q^{1-g}\frac{q-1}{h}G(t).
\end{aligned}$$

Therefore,

$$G(1/qt) = t^{2-2g}q^{1-g}G(t). \tag{5}$$

Next are going to compute $F(1/qt)$. We have

$$F(1/qt) = \sum_{m=0}^{2g-2} \sum_{[D]\in \mathrm{Pic}^m(X)} \frac{q^{\ell([D])}}{q-1}(qt)^{-m}.$$

In view of Theorem 1.2, we have that the map

$$\mathrm{Pic}^m(X) \to \mathrm{Pic}^{2g-2-m}$$
$$[D] \mapsto [\mathcal{K} - D],$$

is a bijection. Moreover, as m runs through $\{0, \cdots, 2g-2\}$ so does $2g-2-m$. Thus, the sum can be rewritten as

$$F(1/qt) = \sum_{m=0}^{2g-2} \sum_{[D]\in \mathrm{Pic}^m(X)} \frac{q^{\ell([\mathcal{K}-D])}}{q-1}(qt)^{m+2-2g}.$$

Furthermore, Theorem 1.2 yields that $\ell([\mathcal{K} - D]) = \ell([D]) - (\deg([D]) - g + 1)$. Therefore, we get

$$\begin{aligned} F(1/qt) &= \sum_{m=0}^{2g-2} \sum_{[D]\in \mathrm{Pic}^m(X)} \frac{q^{\ell([D])-m+g-1}}{q-1}(qt)^{m+2-2g} \\ &= t^{2-2g}q^{1-g} \sum_{m=0}^{2g-2} \sum_{[D]\in \mathrm{Pic}^m(X)} \frac{q^{\ell([D])}}{q-1}t^m \\ &= t^{2-2g}q^{1-g}F(t). \end{aligned}$$

Thus,

$$F(1/qt) = t^{2-2g}q^{1-g}F(t). \tag{6}$$

Combining (5) and (6) we get

$$Z(X, 1/qt) = q^{1-g}t^{2-2g}Z(X, t).$$

The theorem follows. □

References

[Must] Mircea Mustata, *Zeta functions in algebraic geometry*, lecture notes.

[Popa] Mihnea Popa, *Modern aspects of the cohomological study of varieties*, lecture notes.

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