Lior Silberman's Math 322: Problem Set 11 (not for submission)

Nilpotent groups

1. Fix a field F (say $F=\mathbb{R}$) and $n\geq 2$. Let $U_n\subset \mathrm{GL}_n(F)$ denote the group of upper-triangular matrices with 1s on the main diagonal. Write E^{ij} for the matrix having 1 at position ij and 0 everywhere else.
   (a) Show that $Z(U_n)=U_{n,n-1}=\left\{I_n+zE^{1,n}\mid z\in F\right\}$, the matrices whose only non-zero entry (above the main diagonal) is in the upper right corner.
   (b) Show that the equivalence class of $u\in U_n$ in $U_n/Z(U_n)$ depends exactly on the entries of u_n except the corner one.
   (c) Show that $U_{n,n-2}=\left\{u\in U_n\mid 2\leq j<i+n-2\to u_{ij}=0\right\}=\left\{I_n+\sum_{j-i\geq n-2}z_{ij}E^{ij}\right\}$ is the subgroup $Z^2(U_n)\lhd U_n$ which contains $Z(U_n)$ and such that $Z^2(U_n)/Z(U_n)$ is the center of $U_n/Z(U_n)$.
   (d) For each $1\leq m\leq n-1$ let
   $$U_{n,m}=\left\{u\in U_n\mid 2\leq j<i+m\to u_{ij}=0\right\}=\left\{I_n+\sum_{j-i\geq m}z_{ij}E^{ij}\mid z_{ij}\in F\right\}$$
   be the group with non-zero entries starting in the mth diagonal above the main diagonal (note that $U_{n,1}=U_n$). Show that $U_{n,m}$ is normal in U_n.
   (e) Show that $U_{n,m}/U_{n,m+1}$ is the center of $U_n/U_{n,m+1}$ and conclude that $Z^i(U_n)=U_{n,n-i}$ and that U_n is nilpotent.

DEFINITION. For $A,B\subset G$ write $[A,B]$ for the subgroup $\langle\{[a,b]\mid a\in A,b\in B\}\rangle$ generated by all commutators of elements from A,B.

2. (Descending central series) Let $\gamma_1(G)=G$ and $\gamma_{i+1}(G)=[G,\gamma_i(G)]$.
   (a) Show by induction that $\gamma_i(G)$ are normal subgroups such that $\gamma_{i+1}(G)\subset\gamma_i(G)$.
   (b) Show that $\gamma_i(G)/\gamma_{i+1}(G)$ is contained in the centre of $G/\gamma_{i+1}(G)$.
   (c) Suppose G is nilpotent of degree d, so that $Z^d(G)=G$. Show that $\gamma_i(G)\subset Z^{d+1-i}(G)$.

Solvable groups

3. Let G be a group of order p^aq^b. In each case show that G is solvable (hint: find a normal subgroup N and consider N and G/N separately).
   (a) $a=2,b=1$.
   (b) $a=2,b=2$.

4. Let $n\geq 2$ and let $B_n(F)\subset\mathrm{GL}_n(F)$ be the subgroup of upper-triangular invertible matrices.
   (a) Show that $U_n\lhd B_n$ and that $B_n/U_n\simeq\left(F^\times\right)^n$.
   (b) Show that $B_n(F)$ is solvable.
   (c) Show that (unless $F=\mathbb{F}_2$) $Z(B_n(F))$ consists exactly of the scalar matrices with non-zero entries.
   (d) Show that for a large enough field F, $Z(B_n/Z(B_n))=\{e\}$. Conclude that B_n is solvable but not nilpotent (this holds for any $F\neq\mathbb{F}_2$).

The derived series

5. Fix a group G. The subgroup $G' = [G,G]$ is called the *derived subgroup*.
   (a) Show that G' is a normal subgroup of G.
   (b) Let $N \lhd G$. Show that G/N is abelian iff $G' \subset N$.

   DEF The descending series of subgroups defined by $G^{(0)} = G$ and $G^{(i+1)} = \left(G^{(i)}\right)'$ is called the *derived series*.

   (c) Show that $G^{(i)}/G^{(i+1)}$ is abelian.

6. Let $G = G_0 \rhd G_1 \rhd G_2 \cdots \rhd G_k$ be a descending series of subgroups of G with G_{i-1}/G_i abelian. Note that we don't assume $G_k = \{e\}$.
   (a) Writing $G^{(1)} = G'$ show that $G^{(1)} \subset G_1$.
   (b) Writing $G^{(2)} = (G')'$ show that $G^{(2)} \subset G_1' \subset G_2$.
   (c) Writing $G^{(i+1)} = \left(G^{(i)}\right)'$ show by induction that $G^{(i)} \subset G_i$ for each i.
   (d) Show that G is solvable iff $G^{(n)} = \{e\}$ for some n.

Supplementary problems

Let G be a nilpotent group.

A. Show that X generates G iff its image generates $G^{\mathrm{ab}} = G/G'$.

B. Suppose G is finitely generated. Show that every subgroup of G is finitely generated.

B. Show that G_{tors} is a subgroup of G.