Math 322, lecture 11, 12/10/17

Last time: (1) Isom thms ← PS5: $H/_{H\cap N} \cong HN/_N$ if $H\cap N = \{e\}$

(2) A_n simple ($n\geq 5$)

Today: (1) A_n simple ($n\geq 5$)

(2) Group actions

Thm: Let $n\geq 5$ then A_n is simple.

Pf: Let $N \triangleleft A_n$ be normal, non-trivial.

Then there exists $\sigma \in N\setminus\{e\}$ of minimal support.

wlog $\text{supp}(\sigma) = \{1, 2, \ldots, k\}$.

(1) $k\neq 1$ ($\sigma \neq id$)

(2) $k\neq 2$ (transpositions are odd)

(3) If $k=3$, $\sigma = 3$-cycle, then N contains all 3-cycles

~~Clemma~~ (lemma: if σ, σ' 3-cycles $\exists \rho \in A_n: \rho\sigma\rho^{-1} = \sigma'$) + N is normal

then $A_n \supseteq N \supseteq \langle 3\text{-cycles}\rangle = A_n$

(4) If $k=4$, $\sigma = 4$-cycle or pdt of two 2-cycles

but 4-cycles are odd, so σ = pdt of two disjoint transpositions. By lemma all of those are conjugate in A_n, so N contains all of them, since they together generate A_n, $N = A_n$

(5) $k \geq 5$, and σ has a cycle of length ≥ 3

wlog $\sigma(1) = 2$, $\sigma(2) = 3$, σ moves $4, 5$ as well

let $\gamma = (345)\sigma(345)^{-1}\sigma^{-1}$.

if $\sigma(i) = i$ then $i > 5$. Then $\sigma^{-1}(i) = i$ as well

also (345), and $(345)^{-1} = (543)$ only move $3, 4, 5$ not i.

It follows that $\gamma(i) = i$.

Also, (345) fixes 2, $(345)^{-1}$ fixes 1, so $\gamma(2) = 2$.

Now $(345)\sigma(345)^{-1} \in N$ since N is normal,

so $\gamma \in N$ has smaller support that σ.

and $\sigma^{-1} \in N$

But $\gamma(3) = 4$ so $\gamma \neq id$, a contradiction

(6) $k \geq 5$, σ pdt of disjoint transposition, wlog

$\sigma = (12)(34)(56)(78) \cdots$ (at least 4 such since σ is even)

Define γ same way: $\gamma = (345)\sigma(345)^{-1}\sigma^{-1}$. Again $\gamma \in N$

Again γ fixes every fixed pt of σ and also $1, 2$.

But $\gamma(7) = 8$, $\gamma(8) = 7$ so $\gamma \neq id$, again a contradiction.

$\boxed{N}$

Chapter 3: Group Actions

Def: An action of the group G on the set X is a binary operation $\cdot : G \times X \to X$ s.t.:

(1) $e_G \cdot x = x$ for all $x \in X$

(2) $h \cdot (g \cdot x) = (hg) \cdot x$ for all $h, g \in G$, $x \in X$.

Def: A G-set is a pair $(X, \cdot)$ where X is a set, $\cdot$ is an action of G on X.

sometimes write $G \circlearrowright X$

Examples: (0) trivial action: any G, X set $g \cdot x = x$

(1) Key example: S_X acts on X by evaluation:

$$\sigma \cdot i \overset{def}{=} \sigma(i)$$

(def of $\cdot$ in S_X was that $(\sigma \circ \tau)(i) \overset{def}{=} \sigma(\tau(i))$

(2) F field, V vector space $/F$, then scalar multiplication is an action $F^\times \circlearrowright V$

(also $GL(V) \circlearrowright V$)

$$T \cdot \underline{v} = T(\underline{v})$$

(3) X set with "structure", $\mathrm{Aut}(X) = \{\sigma \in S_X \mid \sigma, \sigma^{-1}$ "preserve the structure"$\}$

- Eg. V vsp, G = invertible maps
- D_{2n} acting on X = vertices of  $\leftarrow C_n$

(4) G gp, $\mathrm{Aut}(G) = \{ f \in \mathrm{Hom}(G,G) \mid f \text{ is an isom} \}$ acts on G.

P56: Induced actions: say G acts on X, Y.

$\Rightarrow$ G acts on functions from X to Y.

$\Rightarrow$ G acts on subsets of X: $g \cdot A = \{ g \cdot a \}_{a \in A}$ if $A \subset X$.

$\vdots$

* Regular action

Claim: Left multiplication is an action of G on itself:

set $g \cdot x = gx$

New point of view: fix $g \in G$ define $\sigma_g : G \to G$ by

$$\sigma_g(x) = gx$$

~~Lemma: (1) $\sigma_g \in S_G$ for~~

More generally, let G act on X, define $\sigma_g : X \to X$ by $\sigma_g(x) = g \cdot x$

Lemma: (1) $\sigma_g \in S_X$ for all $g \in G$.

(2) The map $g \mapsto \sigma_g$ is a hom $G \to S_X$.

(3) The map $\left\{ \begin{matrix} \text{actions of} \\ G \text{ on } X \end{matrix} \right\} \to \mathrm{Hom}(G, S_X)$

$\mapsto \sigma.$

is a bijection

Pf: First verify that $\sigma_g \circ \sigma_h = \sigma_{gh}$.

Indeed: $(\sigma_g \circ \sigma_h)(x) \overset{\text{def of } \circ}{=} \sigma_g(\sigma_h(x)) = \sigma_g(h \cdot x) \overset{\text{def of } \sigma_g, \sigma_h}{=} g \cdot (h \cdot x)$

$= (gh) \cdot (x) = \sigma_{gh}(x)$

$\uparrow$ an action

(1) By definition of action, $\sigma_{e_G} = id_X$.

Now for any $g \in G$, $\sigma_g \circ \sigma_{g^{-1}} = \sigma_{gg^{-1}} = \sigma_{e_G} = id_X$

$\sigma_{g^{-1}} \circ \sigma_g = \sigma_{g^{-1}g} = \sigma_{e_G} = id_X$

so σ_g is a bijection, ~~as~~ i.e. $\sigma_g \in S_X$.

(2) We already checked $\sigma_g \circ \sigma_h = \sigma_{gh}$.

(3) ~~Have~~ map $\{\text{actions}\} \to \{\text{homs}\}$

need inverse. Given $\sigma \in \mathrm{Hom}(G, S_X)$ define an action of G on X by $g \cdot x = (\sigma(g))(x)$

Indeed $e_G \cdot x = (\sigma(e_G))(x) = id_X(x) = x$.

$$g \cdot (h \cdot x) \overset{\text{def}}{=} (\sigma(g))\big((\sigma(h))(x)\big) \underset{\text{def of } \circ}{=} (\sigma(g) \circ \sigma(h))(x) \underset{\sigma \text{ is a hom}}{=} (\sigma(gh))(x) =$$

$$= (gh) \cdot x.$$

Clear this is indeed the inverse map. $\blacksquare$

<u>Remark 1</u>: I will not distinguish actions of G on X and homs $G \to S_X$.

<u>Remark 2</u>: This lemma is an important source of homomorphisms, and of normal subgps (kernels)

1st Payoff:

<u>Thm</u>: (Cayley 1878): Every group G is isomorphic to a subgp of S_G. In particular, every gp of order n is isom to a subgp of S_n.

<u>Pf</u>: Consider the left-regular action of G on itself.
This gives a hom $L_G : G \to S_G$

$$(L_G(g))(x) = gx$$

Say $g \in \mathrm{Ker}(L_G)$ then $L_G(g) = id_G$ i.e. $g \cdot x = x$ for all $x \in G$ so $g = e_G$, i.e. $\mathrm{Ker}\, L_G = \{e_G\}$ and L_G is injective, i.e. an isom onto its image $\square$

<u>Example</u>: Let p be prime. Then C_p embeds in S_n iff $p \leq n$.

<u>Pf</u>: If $n \geq p$, S_n contains a p-cycle, and if C_p embeds in S_n then by Lagrange $p \mid n!$ but if $n < p$ all prime factors of $n!$ are less than p so $p \nmid n!$ and no embedding is possible