Lior Silberman's Math 535, Problem Set 4: Preliminaries on Tori

Connected abelian Lie groups

1. Let $\Lambda < \mathbb{R}^d$ be a discrete subgroup. Show that $\Lambda = \oplus_{i=1}^{k} \mathbb{Z}\underline{v}_i$ for a linearly independent set $\{\underline{v}_i\}_{i=1}^{k} \subset \mathbb{R}^d$. Conversely show that such a subgroup is discrete.

2. Let G be an Abelian Lie group, and suppose that $\pi_0(G) = G/G^\circ$ is finite. Show that $G \simeq G^\circ \times \pi_0(G)$. (Hint: show that a connected abelian Lie group is divisible).

Tori

3. (Fourier analysis on tori) Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-torus. A *trigonometric polynomial* on $\mathbb{T}^n$ is a function of the form $f(\underline{x}) = \sum_{i=1}^{I} a_i e\left(\underline{k}_i \cdot \underline{x}\right)$ where $\underline{k}_i \in (\mathbb{Z}^n)^*$ lie in the dual lattice.
   (a) Use Peter–Weyl to show that the space of trigonometric polynoimals is dense in $C\left(\mathbb{T}^n\right)$ and $L^2\left(\mathbb{T}^n\right)$.
   (b) Use Stone–Weierstrass instead to show that the trigonometric polynomials are dense in $C\left(\mathbb{T}^n\right)$, and use that to show that their orthocomplement in $L^2\left(\mathbb{T}^n\right)$ vanishes, getting density there too.
   (c) For $f \in L^2(\mathbb{T}^n)$ and $\underline{k} \in (\mathbb{Z}^n)^*$ set $\hat{f}\left(\underline{k}\right) = \int_{\mathbb{T}^n} f(\underline{x}) e\left(-\underline{k} \cdot \underline{x}\right) d^n x$ (probability Haar measure). Then $\sum_{\underline{k}} \hat{f}(\underline{k}) e(\underline{k} \cdot \underline{x})$ converges in L^2 to f.
   (d) For $f \in C^m(\mathbb{T}^n)$ use integration by parts to show that $\left|\hat{f}(\underline{k})\right| \leq C_f\left(1 + |\underline{k}|\right)^{-m}$. Conclude that for $m > n$, the series $\sum_{\underline{k}} \hat{f}(\underline{k}) e(\underline{k} \cdot \underline{x})$ converges in C^{m-n-1} to f.
   (e) (Weyl criterion) Let $\left\{\mu_j\right\}_{j=1}^{\infty}$ be a sequence of Borel probability measures on $\mathbb{T}^n$. Show that $\mu_j(f) \to \mu(f)$ for every f iff this holds for the *plane waves* $f(\underline{x}) = e\left(\underline{k} \cdot \underline{x}\right)$
4. (Weyl equidistribution) Let $\{\xi_i\}_{i=0}^{n} \subset \mathbb{R}$ be linearly independent over $\mathbb{Q}$ where $\theta_0 = 1$, and let $\underline{\xi} = (\xi_i)_{i=1}^{n} \mod \mathbb{Z}^n \in \mathbb{T}^n$. Show that the sequence $\left\{k\underline{\xi}\right\}_{k=1}^{\infty} \subset \mathbb{T}^n$ is *uniformly distributed*: for any open $U \subset \mathbb{T}^n$,
$$\frac{1}{K}\#\left\{1 \leq k \leq K \mid k\underline{\xi} \in U\right\} = \frac{\mathrm{vol}(U)}{\mathrm{vol}(\mathbb{T}^n)}.$$
   Conclude that the sequence $\left\{k\underline{\xi}\right\}_{k=1}^{\infty}$ is *dense* in the torus.
   Hint: Let $\mu_K = \frac{1}{K}\sum_{k=1}^{K} \delta_{k\underline{\xi}}$. By 1(e) to show $\mu_K \xrightarrow[K\to\infty]{\text{wk-*}} \mathrm{vol}$ it suffices to test against plane waves.