Lior Silberman's Math 535, Problem Set 5: Compact Lie Groups

The classical groups

1. Find a maximal torus and the Weyl group of $\mathrm{SU}(n)$, $\mathrm{SO}(2n)$, $\mathrm{SO}(2n+1)$.

2. (More on covering groups)
   (a) Show that $\mathrm{U}(n) \simeq (\mathrm{SU}(n) \times \mathrm{U}(1)) / \mu_n$ where $\mu_n \subset \mathrm{U}(1)$ is the group of nth roots of unity.
   (b) Show that $\mathrm{U}(n)$ is not isomorphic to $\mathrm{SU}(n) \times \mathrm{U}(1)$ (this is less obvious than it seems).

3. (Symplectic groups over fields)
   DEF Let F be a field, $\mathrm{char} F \neq 2$, V a finite-dimensional F-vector space. A *symplectic form* on V is a non-degenerate anti-symmetric bilinear form $\langle \cdot, \cdot \rangle$ on V.
   (a) (Darboux's Theorem) Show that there is a basis $\{e_i\}_{i=1}^n \cup \{f_i\}_{i=1}^n$ such that $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$ and such that $\langle e_i, f_j \rangle = \delta_{ij}$. In particular, $\dim_F V$ is even.
   (b) The *symplectic group* is the associated symmetry group

$$\mathrm{Sp}_{\langle \cdot, \cdot \rangle}(F) = \{ g \in \mathrm{GL}(V) \mid \forall \underline{u}, \underline{v} \in V : \langle g\underline{u}, g\underline{v} \rangle = \langle \underline{u}, \underline{v} \rangle \} .$$

   Show that up to conjugacy this group does not depend on the choice of symplectic form.
   (c) Given $\underline{u} \in V$ and $a \in F$, a *symplectic transvection* is the map $U_{\underline{u},a}(\underline{v}) = \underline{v} + a \langle \underline{v}, \underline{u} \rangle \underline{u}$. Show that $U_{\underline{u},a} \in \mathrm{Sp}_{\langle \cdot, \cdot \rangle}(F)$.
   (d) Show that the representation of the symplectic group of V on V is irreducible.
   DEF Write $\mathrm{Sp}_{2n}(F)$ for the symplectic group with respect to the *standard form*: $\mathrm{Sp}_{2n}(F) = \left\{ g \in \mathrm{GL}_{2n}(F) \mid g^T X g = X \right\}$ where $X = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ (I_n is the $n \times n$ identity matrix).

4. (The compact symplectic group) This is the group $\mathrm{Sp}(n) \stackrel{\text{def}}{=} \mathrm{Sp}_{2n}(\mathbb{C}) \cap \mathrm{U}(2n)$. Equivalently, we endow a complex symplectic vector space with the Hermitian product for which the symplectic basis of 3(a) is orthonormal.
   (a) Find a maximal torus of $\mathrm{Sp}(n)$, and its associated Weyl group.
   (b) Show that the normalizer of the torus acts irreducibly on $\mathbb{C}^{2n}$.
   (c) Show that $\mathrm{Sp}(n)$ is a maximal compact subgroup of $\mathrm{Sp}_{2n}(\mathbb{C})$.

Roots and root spaces

5. Let $\alpha, \beta \in \Phi(G : T)$ be non-proportional roots, let $I = \{ k \in \mathbb{Z} \mid \beta + k\alpha \in \Phi \}$, and let $a = \min I$, $b = \max I$. Show that:
   (a) $I = [a, b] \cap \mathbb{Z}$.
   DEF Call $\{\beta + k\alpha\}_{k \in I}$ a "root string", specifically the *α-string through β*.
   (b) $\oplus_{k \in I} \mathfrak{g}_{\beta + k\alpha}$ is invariant by $\mathrm{ad} X_\alpha, \mathrm{ad} X_{-\alpha}, \mathrm{ad} H_\alpha$, hence by s_α.
   (c) s_α acts on the root string by reversing the order; in particular $s_\alpha(\beta + a\alpha) = \beta + b\alpha$.
   (d) $a + b = -n_{\alpha\beta}$ and the string contains at most 4 elements.
   Hint: apply Corollary 153.

6. Let $\mathfrak{g}$ be a Lie algebra over an arbitrary field. For $X,Y \in \mathfrak{g}$ set $\langle X,Y\rangle \stackrel{\text{def}}{=} \operatorname{Tr}(\operatorname{ad}_X \operatorname{ad}_Y)$ and call this the *Killing form* of $\mathfrak{g}$.
   (a) Show that the Killing form is bilinear and symmetric.
   (b) Show that it is ad-invariant: $\langle \operatorname{ad}_Z X, Y\rangle + \langle X, \operatorname{ad}_Z Y\rangle$.
   (c) Show that the radical of the Killing form is an ideal of $\mathfrak{g}$ containing the centre.
   (d) Suppose $\mathfrak{g}$ is the Lie algebra of a real Lie group G. Show that the Killing form is *Ad*-invariant: $\left\langle \operatorname{Ad}_g X, Y\right\rangle = \left\langle X, \operatorname{Ad}_g Y\right\rangle$.
   (e) Suppose further G is a compact Lie group. Show that the Killing form is negative semi-definite and that its radical is the center.
   (f) Conversely, suppose the Killing form of a Lie group G is negative definite. Show that the centre of G is discrete, and that G is compact as long as its centre is finite.

From hyperplane arrangements to Weyl chambers

Let E be a finite-dimensional affine vector over $\mathbb{R}$. A *hyperplane* in E is an affine subspace $H \subset E$ of codimension 1. The complement $E \setminus H$ has two connected components, the *half-planes* bounded by H. Both are covex sets.

A *hyperplane arrangement* is a set $\mathcal{H}$ of hyperplanes. Call $\mathcal{H}$ *locally finite* if every $x \in E$ has a neighbourhood intersecting only finitely many $H \in \mathcal{H}$.

7. (Facets) Fix a locally finite hyperplane arrangement $\mathcal{H}$ in E.
   (a) Define a relation on E by $x \sim y$ if for every $H \in \mathcal{H}$ either both $x,y \in H$ or the closed interval $[x,y]$ is disjoint from H (that is, if both x,y are on the same side of H). Show that $\sim$ is an equivalence relation.
   DEF Equivalence classes are called *facets*; for $x \in E$ write $F(x)$ for the facet containing it.
   (b) Show that the facets are convex.
   (c) Show that for every $x \in E$ has an open neighbourhood U such that every hyperplane intersecting U passes through x.
   (d) Show that $F(x)$ is open in $\bigcap\{H \in \mathcal{H} \mid x \in H\}$. Conclude that this affine subspace is the affine hull of $F(x)$ and that $F(x)$ is open in its closure.
   DEF For a facet F write $\dim F$ for the dimension of its affine hull.
   (e) Show that the complement of all the hyperplanes is an open dense subset of E, and that its connected components are exactly the facets of dimension $\dim E$.
   DEF Call these facets of maximal dimension *chambers*.
   (f) For any facet F show that ∂F is a union of facets of strictly smaller dimension.

8. (Walls) Partially order the facets by setting $F \geq F'$ if $F' \subset \bar{F}$.
   (a) Suppose $F \geq F'$ and let H be a hyperplane not containing F'. Show that both F, F' are on the same side of H.
   (b) Suppose $F \geq F'$ and that $\dim F \geq \dim F' + 2$. Show that there is a facet F'' such that $F \geq F'' \geq F'$ and $\dim F'' = \dim F + 1$. (hint: consider the hyperplanes containing F' but not F and remove them one-by-one).
   DEF Call $H \in \mathcal{H}$a *wall* of the chamber C if some codimension-1 facet of C is open in H.
   (c) Show that ∂C is covered by the sets $H \cap \bar{C}$ where H is a wall of C.
   (d) Show that every $H \in \mathcal{H}$ is the wall of some chamber (hint: find $x \in H$ which does not lie in any other hyperplane and choose C such that $x \in \bar{C}$).

9. (Reflection groups) Suppose now that E is a Euclidean space (that is, it is equipped with a Euclidean norm), and for each $H \in \mathcal{H}$ let $s_H \in \mathrm{Isom}(E)$ be the orthogonal reflection in H. Suppose that $\mathcal{H}$ is s_H-invariant for each $H \in \mathcal{H}$ (that is, if $H, H' \in \mathcal{H}$ then $s_H(H') \in \mathcal{H}$ as well).
   (a) When $\dim E = 2$, let $\ell_1, \ell_2 \subset E$ be two distinct intersecting lines and let s_i be the reflection in ℓ_i. Show that $s_1 s_2$ is a rotation by 2θ about the intersection point where $0 < \theta \leq \frac{\pi}{2}$ is the angle between ℓ_1, ℓ_2.
   (b) Using the assumption that $\mathcal{H}$ is locally finite show that $s_1 s_2$ has finite order and hence that θ is a rational multiple of π for some $m \geq 2$. Show that s_1, s_2 commute iff $m = 2$ iff $\ell_1 \perp \ell_2$.
   (c) Suppose that ℓ_1, ℓ_2 are both walls of the same chamber. Show that $\theta = \frac{\pi}{m}$ for some $m \geq 2$ and that the order of $s_1 s_2$ is exactly m.
   (d) Now let $\dim E \geq 2$ be arbitrary and let $H_1, H_2 \in \mathcal{H}$ be distinct non-parallel hyperplanes and let s_i be the associated reflections. Applying the ideas of (b),(c),(d) in the orthogonal complement to $H_1 \cap H_2$ show that $s_1 s_2$ has finite order, that the angle between H_1, H_2 is rational, and that if H_1, H_2 are walls of the same chamber then the angle between them is $\frac{\theta}{m}$ for some $m \geq 2$.
   (e) Let H_1, H_2 be distinct parallel hyperplanes. Show that $s_{H_1} s_{H_2}$ is a translation in the direction perpendicular to them, and in particular that it has infinite order.

10. (Weyl groups and coxeter groups) Continuing with the hypothesis of the previous problem, let $W \subset \mathrm{Isom}(E)$ be the subgroup generated by the reflections $\{s_H\}_{H \in \mathcal{H}}$ and let $W' \subset W$ be the subgroup generated by the reflection in the wall of a fixed chamber C.
   (a) Show that W' acts transitively on the set of chambers
   (b) Show that $W' = W$.
   DEF Let $\{H_i\}_{i \in I}$ be the walls of C, and let $s_i \in W$ be the reflection by H_i. For $i \neq j$ let the angle between H_i, H_j be $\frac{\pi}{m_{ij}}$ (so that $m_{ij} = \infty$ if H_i, H_j are parallel). If $i = j$ set $m_{ij} = 1$. The matrix M is called the *Coxeter matrix* of W.
   DEF A *Coxeter matrix* (of rank n) is an $n \times n$ matrix M such that $m_{ii} = 1$ and so that for $i \neq j$ we have $m_{ij} = m_{ji} \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$.
   (c) The *Coxeter group* associated to a Coxeter matrix M is the group $W(M)$ generated by $S = \{s_i\}_{i=1}^{n}$ subject to the relations $\left(s_i s_j\right)^{m_{ij}} = 1$ for all i, j. Show that W is a quotient of $W(M)$.
   RMK In fact, $W = W(M)$.

REMARK. Finite Coxeter groups can be classified.